

Point-Set Topology/Geometry/Ag1 Topology Select Solutions

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1. Nested closed subsets of compact

Let (X, τ) be a compact topological space. Let $E_1 \supset E_2 \supset \dots$ be closed subsets of X . If X is compact, prove that $\bigcap_{i \in I} E_i \neq \emptyset$.

Proof. Let us assume toward a contradiction that

$$\bigcap_{n \in \mathbb{N}} E_n = \emptyset$$

Since for each $n \in \mathbb{N}$ we know E_n is closed in X and they are decreasing, we know the $X \setminus E_n$ are increasing. Then we have that for each n ,

$$X \subseteq \bigcup_{n \in \mathbb{N}} X \setminus E_n.$$

By compactness of X we have the existence of a finite subset $A \subset \mathbb{N}$ such that

$$\begin{aligned} X &\subseteq \bigcup_{n \in A} X \setminus E_n \\ &= X \setminus E_M, \end{aligned}$$

Where $M \in A$ is the maximal element in terms of set containment. But this implies

$$E_n = \emptyset$$

For every n such that $1 \leq n < M$, contradicting non-emptiness. □

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2 Compactness

(a) Closed subspaces of compact topological spaces need be compact.

Proof. Let X be a compact topological space. Suppose $Y \subseteq X$ is closed. We would like to show that Y is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of Y . That is,

$$Y \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

As $Y \subseteq X$ is closed, its complement in X , $X \setminus Y$, is open in X . Moreover, the union of the U_α together with $X \setminus Y$ forms an open cover for X , i.e.,

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha \cup X \setminus Y.$$

Since X is compact, we know there exists a finite subset $B \subseteq A$ such that

$$X \subseteq \bigcup_{i \in B} U_{\alpha_i} \cup X \setminus Y$$

And since $Y \subseteq X$ we have that

$$Y \subseteq \bigcup_{i \in B} U_{\alpha_i}$$

thus we have found a finite subcover of $\{U_\alpha\}$ that cover Y and so Y is compact as needed. \square

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(b) Compact subspaces of Hausdorff topological spaces need be closed.

Proof. Let X be a Hausdorff topological space. Suppose $Y \subseteq X$ is compact. We wish to show Y is closed in X . It suffices to show its complement $X \setminus Y$ is open in X . Let $x \in X \setminus Y$, as X is Hausdorff, for each $y \in Y$ there exists open sets $U_y \subseteq X \setminus Y$, $V_y \subseteq Y$ with $x \in U_y$, $y \in V_y$ such that

$$U_y \cap V_y = \emptyset.$$

Since $V_y \subseteq Y$ is open, $\{V_y \mid y \in Y\}$ is an open cover for Y , that is,

$$Y \subseteq \bigcup_{y \in Y} V_y.$$

By compactness of Y there exists a finite subset $A \subseteq Y$ such that

$$Y \subseteq \bigcup_{y \in A} V_y = V.$$

Then the finite intersection $U = \bigcap_{y \in A} U_y$ is an open neighborhood of x disjoint from V , namely

$$x \in U \subseteq X \setminus Y$$

and thus $X \setminus Y$ is open in X forcing $Y \subseteq X$ to be closed as needed. \square

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(c) The image of a compact topological space need be compact under a continuous map.

Proof. Let $f : X \rightarrow Y$ be a continuous map between topological spaces. Suppose X is compact. We would like to show that $f(X) \subseteq Y$ is compact. Let $\{V_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for $f(X)$. That is,

$$f(X) \subseteq \bigcup_{\alpha \in A} V_\alpha$$

where $V_\alpha \subseteq Y$ is open for each $\alpha \in A$. As f is continuous, we have that

$$f^{-1}(V_\alpha) \subseteq X$$

is open for each $\alpha \in A$ as well. Then $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ is an open cover for X and since X is compact we have the existence of a finite subset $B \subseteq A$ such that

$$X \subseteq \bigcup_{b \in B} f^{-1}(V_b)$$

then it follows that

$$f(X) \subseteq \bigcup_{b \in B} V_b$$

thus $f(X)$ is compact. □

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3. Regular space equivalence

(a) Define a regular space.

Proof. A topological space X is called *regular* or T_3 if $\forall x \in X$ and any closed subset $F \subset X$ not containing x , $\exists U, V \subset X$ open with $x \in U, F \subset V \ni$

$$U \cap V = \emptyset.$$

Namely, we can separate points from closed sets using open sets. □

Assume X is Hausdorff. Then X is regular if and only if for every $x \in X$ with neighborhood $U \subseteq X$ there exists $V \subseteq X$ open with $x \in V$ such that $\bar{V} \subset U$.

Proof. Let X be a Hausdorff topological space. Suppose first that X is also regular. Let $x \in X$ and let $U_x \subseteq X$ be an open neighborhood of x . Then $F = X \setminus U_x$ is closed by definition. Since X is a regular space we can separate x and F with open subsets of X , that is, there exists $V, W \subseteq X$ open with $x \in V, F \subset W$ such that

$$V \cap W = \emptyset.$$

As $x \in U_x$ we have that

$$V \cap F = \emptyset$$

forcing $\bar{V} \subseteq U_x$ as needed.

Next suppose for each $x \in X$ and open neighborhood of x say $U_x \subseteq X$ there exists an open neighborhood $V \subseteq X$ with $x \in V$ such that $\bar{V} \subseteq U_x$. Let $x \in X \setminus F$ where $F \subseteq X$ is closed. We wish to separate these via open subsets of X . As F is closed it follows that $X \setminus F \subseteq X$ is open. Then by our assumption we are guaranteed the existence of an open set $V \subseteq X$ such that

$$\bar{V} \subseteq X \setminus F.$$

Here $x \in V$ is an open neighborhood of x and similarly $X \setminus \bar{V}$ is an open set containing F such that

$$V \cap X \setminus \bar{V} = \emptyset$$

thus X is regular as needed. □

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4. Non-disjoint union of connected is connected

(a) Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of topological spaces. If for each $\alpha \in I$ X_α is connected and

$$\bigcap_{\alpha \in I} X_\alpha \neq \emptyset,$$

then we have that

$$\bigcup_{\alpha \in I} X_\alpha$$

is connected as well.

Note that this fails if we swap the conclusions union with intersection, take $X_1 := S^1, X_2 := \{(x, y) : x = y \in \mathbb{R}\}$, so the circle and the line, their union is connected, their intersection is two disjoint points however.

Proof. First, Recall the Connected lemma:

Let (X, τ) be a topological space. If $A \cup B$ is a separation of the space and $Y \subset X$ is a connected subspace, then Y lies entirely in A or B .

The proof of this is in Exercise 16. Now assume towards a contradiction that $\bigcup_{\alpha \in I} X_\alpha$ is disconnected. That is, there is a separation. I.e.,

$$\bigcup_{\alpha \in I} X_\alpha = A \cup B$$

Where $A, B \in \tau$ are non-empty and disjoint. Since the intersection of the X_α is non-empty, let $x \in \bigcap_{\alpha \in I} X_\alpha$, then $x \in A$ or in B , let us say $x \in A$. But then B is non-empty thus there exists some $y \in B$. But then $y \in X_\beta$ for some $\beta \in I$ and also $x \in X_\beta$ contradicting the connected lemma as X_β is connected it must lie entirely in A or B and so $\bigcup_{\alpha \in I} X_\alpha$ is connected. \square

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(b) The image of a connected space under continuous function need be connected.

Proof. Let $f : X \rightarrow Y$ be a continuous map of topological space. Suppose that X is connected. We wish to show that $f(X)$ is connected as well. Let us suppose towards a contradiction that $f(X)$ has a separation, that is,

$$f(X) = A \cup B$$

where $A, B \subsetneq f(X)$ are both nonempty and open. Since f is continuous, $f^{-1}(A), f^{-1}(B) \subset X$ are both open. Moreover, their union is all of X and thus we have formed a separation of X which is connected, a contradiction and so $f(X)$ is connected. \square

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(c) Let X, Y be connected. Show $X \times Y$ is connected in the product topology.

Proof. By part (a) if we have a union of connected spaces with non empty intersection, then their unions is connected. To fix $y \in Y$. Then for each $x \in X$, the space

$$U_x = (\{x\} \times Y) \cup (X \times \{y\}).$$

It is clear to see that

$$U := \bigcup_{x \in X} U_x = X \times Y.$$

And clearly we have that

$$(\{x\} \times Y) \cap (X \times \{y\}) = (x, y).$$

Furthermore, note that

$$\bigcap U_x = X \times \{y\}.$$

Thus the intersection of the U_x is empty forcing U which is all of $X \times Y$ to be connected. \square

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5. Equivalence for closed in metric space

Let (X, d) be a metric space with $C \subset X$ and some $p \in X$ a point. Prove C is closed if and only if $C \cap \overline{B_R(p)}$ is closed for any $R > 0$ where

$$\overline{B_R(p)} = \{x \in X \mid d(x, p) \leq R\}.$$

Proof. First suppose C is closed. Since arbitrary intersections of closed spaces need be closed we have that the intersection

$$C \cap \overline{B_R(p)}$$

is closed as $\overline{B_R(p)}$ is closed by definition.

On the other hand suppose that for some $p \in X$ and any $R > 0$ the intersection

$$C \cap \overline{B_R(p)}$$

is closed. We wish to show that C is closed, i.e., $C = \overline{C}$. Clearly we have that $C \subseteq \overline{C}$ thus we are left to show that $\overline{C} \subseteq C$. So let $x \in \overline{C}$ be a limit point. We must show $x \in C$. As X is a metric space we can put x in an epsilon ball, that is,

$$x \in B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}.$$

As $C \cap \overline{B_R(p)}$ is closed for any R , we can take $R = d(x, p) + \varepsilon$. Then we have that

$$B_\varepsilon(x) \subset B_R(p) \subset \overline{B_R(p)}.$$

As x is a limit point of C , $B_\varepsilon(x) \setminus \{x\}$ intersects C non-trivially thus x is a limit point of $C \cap \overline{B_R(p)}$ hence $x \in C \cap \overline{B_R(p)}$ forcing $x \in C$ as needed and C is closed. \square

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6. Hausdorff & second-countable

(a) Define *second-countable*

Proof. A topological space X is said to be *second-countable* if it has a countable basis. \square

(b) Define Hausdorff.

Proof. A topological space X is said to be Hausdorff if for any pair of distinct elements $x, y \in X$ we can find open subsets of X say U, V with $x \in U, y \in V$ such that

$$U \cap V = \emptyset.$$

\square

(c) Prove or disprove: Every metric space equipped with the metric topology is Hausdorff.

Proof. Let (X, d) be a metric space. Then d induces a topology on X , namely the collection of epsilon balls, that is,

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}.$$

This collection forms a basis for a topology on X . I claim X is Hausdorff. Let $x, y \in X$ be distinct. Then we have that $d(x, y) > 0$ so denote this distance by ε_0 . Then we have basis elements $x \in B_{\frac{\varepsilon_0}{2}}(x), y \in B_{\frac{\varepsilon_0}{2}}(y)$. It suffices to show

$$B_{\frac{\varepsilon_0}{2}}(x) \cap B_{\frac{\varepsilon_0}{2}}(y) = \emptyset$$

Suppose there exists some $z \in B_{\frac{\varepsilon_0}{2}}(x) \cup B_{\frac{\varepsilon_0}{2}}(y)$ then $d(x, z), d(z, y) < \frac{\varepsilon_0}{2}$. And so by the triangle inequality we have

$$\begin{aligned} \varepsilon_0 &= d(x, y) \\ &\leq d(x, z) + d(z, y) \\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} \\ &= \varepsilon_0 \end{aligned}$$

a contradiction forcing the intersection to be empty as needed thus (X, d) is Hausdorff. \square

(d) Prove or disprove: Every metric space equipped with the metric topology is *second-countable*.

Proof. Consider \mathbb{R} as a metric space equipped with discrete metric, that is

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Here the basis elements consist of singleton sets. \mathbb{R} has an uncountable number of points, the basis (the singletons) is uncountable thus not second-countable. \square

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7. The product topology

Consider the Product space $Y = \prod_{n=1}^{\infty} [0, 1]$ with the product topology.

(a) Prove Y is Hausdorff.

Proof. As $[0, 1]$ is Hausdorff (It is a subspace of the reals which are Hausdorff and it is easy to check subspaces of Hausdorff under subspace topology need be Hausdorff as well) we verify that given a collection $\{(X_i, \tau_i)\}_{i \in I}$ of topological spaces, if they are all assumed to be Hausdorff, then their product

$$(\prod_{i \in I} X_i, \tau_X)$$

(for short we denote $\prod_{i \in I} X_i$ by X) is Hausdorff when τ_X is the product topology. Before proceeding, small lemma:

Lemma If A, B, C are sets such that $B \cap C = \emptyset$, then $A \times B \cap A \times C = \emptyset$ as well and this extends to uncountably infinite case as well.

Now let $x, y \in X$ be distinct. Then there exists (at least one) some $j \in I$ our indexing set such that the components do not agree here, that is, for $x_j, y_j \in X_j$ we have

$$x_j \neq y_j.$$

As X_j is Hausdorff, then there exists $U_j, V_j \in \tau_j$ with $x_j \in U_j, y_j \in V_j$ such that

$$U_j \cap V_j = \emptyset.$$

Then by the definition of the product topology, for each $i \in I \setminus \{j\}$ we can take

$$U_i = V_i = X_i$$

And then define the neighborhoods of x, y as

$$U = \prod_{i \in I} U_i, V = \prod_{i \in I} V_i.$$

and since the U_i, V_i are disjoint at j together with our lemma we have that

$$U \cap V = \emptyset$$

with $x \in U, y \in V$ thus (X, τ_X) is Hausdorff as needed. □

(b) Prove $Y = \prod_{n \in \mathbb{N}} [0, 1]_n$ is separable.

Proof. To show Y is separable, we construct a countable subset that is dense in Y . Consider the following subset of Y ,

$$A := \{(a_1, a_2, \dots) \in Y \mid \exists N \in \mathbb{N} \ni a_i \in \mathbb{Q} \cap [0, 1], 1 \leq i < N\}$$

I.e., all sequences with finitely many rational coordinates. I claim A is dense in Y , to see this we show every open set of Y intersects A nontrivially. Let $x \in U \in \tau_Y$. If $x \in A$ then we are done as

$x \in U \cap A$. Suppose $x \in Y \setminus A$. As we are in the product topology, our basis elements are of the form

$$\prod_{n=1}^{\infty} U_n \subset U$$

Where $U_n = [0, 1]$ for all but finitely many n . And for those finitely many n we have the proper containment $U_n \subset [0, 1]$. If we let $x = (x_1, x_2, \dots)$, then for finitely many values, call them i , we have

$$x_i \in U_i \subset [0, 1]$$

As \mathbb{Q} is dense in \mathbb{R} , we have for these finite many i , there exists some rational $q_i \in \mathbb{Q}$ such that

$$q_i \in U_i$$

Then consider the point $y = (y_1, y_2, \dots)$ such that

$$y_i = q_i; i \leq n, y_n = x_n; \forall n > i$$

Then $y \in A$ and since U was arbitrary we have that A is dense in Y . □

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8. Continuity When restricted

(a) Let (X, τ_X) be a topological space. If we can write $X = \bigcup_{n \in \mathbb{N}} W_n$ where for each $n \in \mathbb{N}$

$$f|_{W_n}: W_n \rightarrow Y$$

is continuous, then

$$f: X \rightarrow Y$$

is continuous.

Proof. Let

$$f: X \rightarrow Y$$

be a map of topological spaces with topologies τ_X, τ_Y respectively. Let $V \in \tau_Y$. Note that

$$\begin{aligned} f|_{W_k}^{-1}(V) &= f^{-1}(V) \cap W_k \\ &\in \tau_X. \end{aligned}$$

As $f|_{W_n}$ is continuous for every n . But then we can write

$$\begin{aligned} f^{-1}(V) &= \bigcup_{k \in \mathbb{N}} f^{-1}(V) \cap W_k \\ &\in \tau_X \end{aligned}$$

as any union of open need be open thus f is continuous. \square

(b) Let (X, τ_X) be a topological space. If $X = A \cup B$ where A, B are closed and $f|_A: A \rightarrow Y, f|_B: B \rightarrow Y$ are continuous. Prove

$$f: X \rightarrow Y$$

is continuous.

Proof. Let $V \subset Y$ be closed, then just as before, since A is a subset of X , then for all subsets of Y which V is, we have

$$f^{-1}(V) \cap A = (f|_A)^{-1}(V)$$

Holds for when $x \in A$ and $f(x) \in V$. As $V \subset Y$ is closed and the restrictions are continuous, we have that

$$\begin{aligned} (f|_A)^{-1}(V) \cap A \\ (f|_B)^{-1}(V) \cap B \end{aligned}$$

Are both closed in A, B respectively thus both are closed in X as well and we have

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \cap (A \cup B) \\ &= (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B) \end{aligned}$$

Which is a finite union of closed thus $f^{-1}(V) \subset X$ is closed as needed thus f is continuous. \square

(c) Assume $X = \bigcup_{k=1}^{\infty} E_k$ where the E_k are all closed in X such that each $E_k \rightarrow Y$ is continuous, is $X \rightarrow Y$ also continuous?

Proof. False. Consider the map

$$f : \mathbb{Z} \rightarrow \mathbb{R}$$

where \mathbb{Z} is endowed with cofinite topology and \mathbb{R} has the standard topology. Let

$$\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} \{n\}$$

So our E_k are just singletons of integers, then $f|_{E_k}$ is continuous for each k .

To see this, take $C \subset \mathbb{R}$ closed, then we have

$$f|_{E_k}^{-1}(C) = f^{-1}(C) \cap E_k$$

which is either just a singleton or the empty set both of which are closed in \mathbb{Z} with cofinite topology.

On the other hand, f is not continuous as any open set $(a, b) \subset \mathbb{R}$ has a pull back with infinite complement thus not open in \mathbb{Z} with cofinite topology. \square

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9. Product Topology

(a) Define the product topology on the product $X = \prod_{j=1}^{\infty} X_j$.

Proof. The product topology on X is the product

$$\prod_{j=1}^{\infty} U_j$$

Where $U_j \subseteq X_j$ are open and

$$U_j = X_j$$

\forall but finitely many j , for finite j ,

$$U_j \subsetneq X_j$$

proper subset. More formally, if π_{β} is projection onto the β th coordinate, then

$$\mathcal{S}_{\beta} = \{\pi_{\beta}^{-1}(U_{\beta}) \mid U_{\beta} \text{ is open in } X_{\beta}\}$$

Then the topology generated by $\bigcup_{\beta \in J} \mathcal{S}_{\beta}$ is the product topology. \square

(b) Show the projection map $p_i : X_1 \times X_2 \rightarrow X_i$ is an open map for $i = 1, 2$.

Proof. For each $i = 1, 2$, let τ_i denote the topology on X_i . Let $U \in \tau_{\alpha}$, then by the definition of the product topology we can write

$$U = \bigcup_{j \in J} \bigcap_{k=1}^{n_j} p_{i_{k,j}}^{-1} U_{k,j},$$

where J is an arbitrary indexing set, $n_j \in \mathbb{N}$ and $i_{k,j} = 1, 2$. Then for every $i = 1, 2$, define $V_{i,k,j} \in \tau_i$ via

$$V_{i,k,j} = \begin{cases} U_{k,j} & ; i = i_{k,j} \\ X_i & ; i \neq i_{k,j} \end{cases}$$

By the definition of projection we have

$$p_{i_{k,j}}^{-1}(U_{k,j}) = V_{1,k,j} \times V_{2,k,j}.$$

And without any loss of generality we can suppose $i = 1$ and compute

$$\begin{aligned} p_1(U) &= \bigcup_{j \in J} p_1\left(\bigcap_{k=1}^{n_j} p_{i_{k,j}}^{-1}(U_{k,j})\right) \\ &= \bigcup_{j \in J} p_1\left(\bigcap_{k=1}^{n_j} (V_{1,k,j} \times V_{2,k,j})\right) \\ &= \bigcup_{j \in J} p_1\left(\bigcap_{k=1}^{n_j} V_{1,k,j} \times \bigcap_{k=1}^{n_j} V_{2,k,j}\right) \\ &= \bigcup_{j \in J} \bigcap_{k=1}^{n_j} V_{1,k,j} \\ &\in \tau_1 \end{aligned}$$

and thus p_1 is an open map. The same proof works for p_2 . \square

(c) If Y is Hausdorff and

$$f : X \rightarrow Y$$

is continuous, prove the graph

$$\Delta = \{(x, f(x)) | x \in X\}$$

is closed in $X \times Y$.

Proof. We show Δ^c is open instead.

Let $(x, y) \in X \times Y \setminus \Delta$, then $y \neq f(x)$.

As $y, f(x) \in Y$ which is Hausdorff they can be separated via open sets of Y .

That is, $\exists U, V \subset Y$ open with $y \in U, f(x) \in V$ s.t.

$$U \cap V = \emptyset$$

By Munkres Theorem 18.1(4) since f is continuous, $\exists W \subseteq X$ with $x \in W$ s.t.

$$f(W) \subseteq V$$

Then $W \times U$ is an open neighborhood of (x, y) disjoint from Δ thus $X \times Y \setminus \Delta = \Delta^c$ is open, and therefore Δ is closed. \square

(d) If Y is Hausdorff and

$$f : X \rightarrow Y$$

is continuous, prove

$$G : X \rightarrow X \times Y$$

defined via

$$G(x) = (x, f(x))$$

is a closed map.

Proof. Let $C \subseteq X$ be closed, we wish to show

$$G(C) \subset X \times Y$$

is closed. Let $(x, y) \in X \times Y \setminus G(C)$, then $y \neq f(x)$ which are both in Y .

Since Y is Hausdorff, $\exists U, V \subseteq Y$ both open with $y \in U, f(x) \in V$ such that

$$U \cap V = \emptyset$$

As f is continuous however, $\exists W \subseteq X$ an open neighborhood of x such that

$$f(W) \subseteq V$$

Then $W \times U$ is an open neighborhood of (x, y) disjoint from $G(C)$, thus $G(C)^c$ is open forcing $G(C)$ to be closed $\therefore G$ is a closed map. \square

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10. Homeomorphisms

(a) Prove $(0,1)$ with the subspace topology is homeomorphic to \mathbb{R} with the standard topology.

Proof. Let

$$f : (0,1) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

be defined via

$$f(x) := \pi x - \frac{\pi}{2}$$

and let

$$g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$$

be defined via

$$g(x) := \tan x$$

Then

$$h : (0,1) \rightarrow \mathbb{R}$$

defined via

$$\begin{aligned} h(x) &:= g(f(x)) \\ &= \tan\left(\pi x - \frac{\pi}{2}\right) \end{aligned}$$

is the desired homeomorphism with inverse defined via

$$h^{-1}(x) := \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2}$$

By calculus we are done. □

(b) Assume X, Y are metric spaces that are homeomorphic. Prove or give counterexample: X complete implies Y complete, that is, completeness preserved under cont?

Proof. Part (a) □

(c) Prove $[a, b] \not\cong (c, d)$.

Proof. First a small lemma (proof left to the interested reader; HINT: First restrict the domain, then restrict the range.)

Lemma: If

$$f : X \rightarrow Y$$

is a continuous map of topological spaces, then for any $x \in X$,

$$\bar{f} : X \setminus \{x\} \rightarrow Y \setminus \{f(x)\}$$

is continuous as well. Moreover, if f is a homeomorphism, then \bar{f} is a homeomorphism as well.

Now let us assume $[a, b] \cong (c, d)$. Then there exists a homeomorphism

$$g : [a, b] \rightarrow (c, d).$$

By our lemma above

$$\bar{g} : [a, b] \setminus \{a\} \rightarrow (c, d) \setminus \{g(a)\}$$

is a homeomorphism as well. I.e.,

$$g : (a, b) \rightarrow (c, g(a)) \cup (g(a), d)$$

is a homeomorphism. Note that the domain is still connected while the range space is clearly disconnected and since connectedness is a topological property, this contradicts continuity of \bar{g} and thus $[a, b] \not\cong (c, d)$ as needed. One can check that the range space in fact has a separation. \square

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11. Topology of finite point-set

Let $X = \{1, 2, 3, 4\}$ be given by the topology $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

(a) Show any two disjoint closed sets have disjoint open neighborhoods

Proof. By definition, the closed sets in X are taken to be the complements of the given open sets, that is,

$$\mathcal{C} = \{X, \emptyset, \{2, 3, 4\}, \{3, 4\}, \{2, 4\}, \{4\}\}$$

are all of the closed subspaces of X . As 4 is an element of every member of \mathcal{C} except \emptyset and \emptyset is disjoint with every non-empty set, the pairs of disjoint closed subspaces are

$$X, \emptyset$$

$$\{2, 3, 4\}, \emptyset$$

$$\{3, 4\}, \emptyset$$

$$\{2, 4\}, \emptyset$$

$$\{4\}, \emptyset$$

where the neighborhood X contains each closed set except for \emptyset and clearly \emptyset is its own neighborhood which is disjoint from X as needed. \square

(b) Show (X, τ) is not T_1

Proof. Consider the elements 1, 2. The open neighborhoods of 2 are

$$\{1, 2\}, \{1, 2, 3\}.$$

Since both of these contain 1 we cannot find open sets for 1 and 2 that do not contain each other thus (X, τ) is not T_1 . \square

(c) Let $A = \{1, 2, 3\} \subset X$ be endowed with the subspace topology. Find disjoint closed subsets of A that do not have disjoint neighborhoods.

Proof. As A is endowed with the subspace topology we can write out the topology on A as follows

$$\tau_A = \{\emptyset, A, \{1\}, \{1, 2\}, \{1, 3\}\}.$$

Then the closed subsets of A are given by

$$\mathcal{C}_A = \{A, \emptyset, \{2, 3\}, \{3\}, \{2\}\}.$$

Then $\{2\}, \{3\}$ are disjoint closed sets in A . The neighborhoods of $\{2\}$ are

$$A, \{1, 2\},$$

and the neighborhoods of $\{3\}$ are

$$A, \{1, 3\}.$$

And since $\{1, 2\} \cap \{1, 3\} = \{1\}$ is non-empty, we have found disjoint closed subsets of A with non disjoint neighborhoods. \square

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12. Closures connected/path

(a) If $X \subseteq Z$ is a connected subset of a topological space, show that $\overline{X} \subseteq Z$ is connected as well.

Proof. Let $X \subseteq Z$ be a connected subspace of a topological space. Suppose towards a contradiction that \overline{X} is not connected. Then there exists a separation of \overline{X} , that is,

$$\overline{X} = A \cup B,$$

$A, B \in \tau_{\overline{X}}$ non-empty and disjoint. As X is connected, by the connected Lemma we have WLOG that $X = A \cap X$. As B is non-empty, it contains some b , namely b is a limit point of X and thus B is an open set containing a limit point of X thus it must intersect $X \subset A$ non-trivially contradicting

$$A \cap B = \emptyset.$$

Thus \overline{X} is connected. □

(b) Show (a) fails for path-connected subspaces.

Proof. Take the topologist's sin curve. That is, the function

$$f : \mathbb{R}^+ \rightarrow [-1, 1]$$

which is defined via

$$x \mapsto \sin\left(\frac{1}{x}\right).$$

Then $\{(x, y) \mid y = \sin(\frac{1}{x})\}$ is path-connected, however the closure given via

$$\{(x, y) \in \mathbb{R}^2 \mid y = \sin\left(\frac{1}{x}\right)\} \cup \{0\} \times [-1, 1]$$

is not path-connected. □

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13. Cofinite Topology

Let τ be the finite complement topology on \mathbb{R} . That is, $U \subseteq \mathbb{R}$ is open if and only if U is empty or $\mathbb{R} \setminus U$ is finite.

(a) Is (\mathbb{R}, τ) Hausdorff?

Proof. I claim (\mathbb{R}, τ) is not Hausdorff. Let $x, y \in \mathbb{R}$ and $U \in \tau$ be a neighborhood of x . Then $\mathbb{R} \setminus U$ is finite so we can write

$$\mathbb{R} \setminus U = \{p_1, \dots, p_n\}.$$

Similarly we can let $V \in \tau$ be a neighborhood of y and write

$$\mathbb{R} \setminus V = \{q_1, \dots, q_m\}.$$

We would like for U, V to have a non-empty intersection. As \mathbb{R} is infinite, we can find a $z \in \mathbb{R}$ such that $z \neq p_i$ for $1 \leq i \leq n$ and $z \neq q_j$ for $1 \leq j \leq m$. This would imply $z \in U \cap V$ and so (\mathbb{R}, τ) is not Hausdorff. \square

(b) Is (\mathbb{R}, τ) compact?

Proof. I claim (\mathbb{R}, τ) is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for \mathbb{R} , that is

$$\mathbb{R} \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

For each $\alpha \in A$ we know $\mathbb{R} \setminus U_\alpha$ is finite, in particular we can take some $\beta \in A$ and so $\mathbb{R} \setminus U_\beta$ is finite. Then for each $x \in \mathbb{R} \setminus U_\beta$ let $U_x \in \tau$ be the neighborhood containing x . Then it follows that

$$\mathbb{R} \subseteq U_\beta \cup \{U_x \mid x \in \mathbb{R} \setminus U_\beta\}$$

is a finite sub-cover of our arbitrary covering thus (\mathbb{R}, τ) is Hausdorff as needed. \square

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14. Sequence of closed

Let F_1, F_2, \dots be a sequence of closed subsets of a topological space X . Suppose that for each $x \in X$ we can find a neighborhood of x say N_x such that $N_x \cap F_j \neq \emptyset$ for only finitely many j values. Prove $\bigcup_{j=1}^{\infty} F_j$ is closed.

Proof. We will show the complement with respect to the entire space is open. That is, we show

$$X \setminus \bigcup_{j=1}^{\infty} F_j$$

Let $x_0 \in X \setminus \bigcup_{j=1}^{\infty} F_j$ be arbitrary. We must find a neighborhood of x that is disjoint from $\bigcup_{j=1}^{\infty} F_j$. As $x_0 \in X$ we are guaranteed the existence of neighborhood N_{x_0} such that

$$N_{x_0} \cap F_j \neq \emptyset$$

for finitely many j values. That is, there exists a finite set J such that

$$N_{x_0} \bigcap_{j \in J} F_j \neq \emptyset.$$

Then I claim the open neighborhood of x_0 that is disjoint from $\bigcup_{j=1}^{\infty} F_j$ is given by the intersection

$$N_{x_0} \bigcap_{j \in J} X \setminus F_j.$$

Clearly we have that $x_0 \in N_{x_0} \bigcap_{j \in J} X \setminus F_j$ and is clearly disjoint from $\bigcup_{j=1}^{\infty} F_j$. Moreover, as τ_X is closed under finite intersecions we have that $N_{x_0} \bigcap_{j \in J} X \setminus F_j \in \tau_X$ as needed for our neighborhood of x_0 and so $X \setminus \bigcup_{j=1}^{\infty} F_j$ is open thus $\bigcup_{j=1}^{\infty} F_j$ is closed. \square

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15 (Incomplete)

Let (X, τ) be a topological space and let \mathcal{C} be the collection of closed sets. A *filter* on \mathcal{C} is a collection \mathcal{F} of sets from \mathcal{C} such that (1) $\emptyset \notin \mathcal{F}$, (2) If $C_1, C_2 \in \mathcal{F}$, then $C_1 \cap C_2 \in \mathcal{F}$, (3) If $C_1 \subset C_2$ with $C_1 \in \mathcal{F}$ and $C_2 \in \mathcal{C}$, then $C_2 \in \mathcal{F}$. Show that if each filter on \mathcal{C} has non-empty intersection, then (X, τ) is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for X . That is,

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha,$$

where $U_\alpha \in \tau$ for each α . Then by definition we have for each $\alpha \in A$ that $X \setminus U_\alpha \in \mathcal{C}$. Let $\mathcal{B} = \{B \in \tau \mid B \subseteq \bigcup_{i \in I} U_{\alpha_i}\}$ for some finite $I \subset A$. Then $X \setminus B \in \mathcal{C}$ for each $B \in \mathcal{B}$. I claim that

$$\mathcal{F}_\beta = \{X \setminus B \mid B \in \mathcal{B}\}$$

defines a filter on \mathcal{C} . For (1), let us suppose $\emptyset \in \mathcal{F}_\beta$, then $\emptyset = X \setminus B$ for some $B \in \mathcal{B}$ which implies $B = X$ and since $B \in \mathcal{B}$, we have that $X \subseteq \bigcup_{i \in I} U_{\alpha_i}$. \square

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16. Connected Lemma

Let (X, τ) be a topological space and $Y \subset X$ a connected subspace endowed with the subspace topology. If $A \cup B$ forms a separation of X , then $Y \subset A$ or $Y \subset B$.

Proof. As $A \cup B$ forms a separation, we have that

$$X = A \cup B$$

where $A, B \in \tau$ are both non-empty and disjoint as a pair. Let us suppose towards a contradiction that there exists $a, b \in Y$ such that $a \in A$ and $b \in B$. I claim then that

$$\begin{aligned} Y \cap X &= Y \cap (A \cup B) \\ &= (Y \cap A) \cup (Y \cap B) \end{aligned}$$

forms a separation of Y . Since Y is endowed with the subspace topology and $A, B \in \tau$ we have that $Y \cap A, Y \cap B \in \tau_Y$. That is, they are both open in Y . If they were not disjoint then there would exist some α such that

$$\alpha \in (Y \cap A) \cap (Y \cap B)$$

contradicting A, B being disjoint as a pair and thus $Y \cap A, Y \cap B$ are disjoint as well. Lastly we know by existence of a, b that they are non-empty thus together they form a separation of Y which is connected a contradiction thus Y must lie entirely within A or B . \square

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17. Arbitrary collection of topologies on a set

(a) Let X be set and $\{\tau_\alpha\}_{\alpha \in A}$ be a collection of topologies on X . Prove $\bigcap_{\alpha \in A} \tau_\alpha$ is a topology on X .

Proof. Suppose for each $\alpha \in A$ we have that (X, τ_α) is a topological space, we must show $(X, \bigcap_{\alpha \in A} \tau_\alpha)$ is also a topological space. First we check $\emptyset, X \in \bigcap_{\alpha \in A} \tau_\alpha$. Since for each $\alpha \in A$, τ_α is a topology on X , we have for every $\alpha \in A$ that $\emptyset, X \in \tau_\alpha$ and thus

$$\emptyset, X \in \bigcap_{\alpha \in A} \tau_\alpha$$

as needed. Next suppose that

$$U_1, U_2, \dots, U_n \in \bigcap_{\alpha \in A} \tau_\alpha.$$

Then for every $\alpha \in A$ we have

$$U_1, U_2, \dots, U_n \in \tau_\alpha.$$

As for each $\alpha \in A$, τ_α is a topology on X , by the closure property we get

$$\bigcap_{i=1}^n U_i \in \bigcap_{\alpha \in A} \tau_\alpha.$$

Lastly we must check arbitrary unions are closed. That is if for every $\beta \in B$ some indexing set let us suppose

$$U_\beta \in \bigcap_{\alpha \in A} \tau_\alpha.$$

Then for every $\alpha \in A$

$$U_\beta \in \tau_\alpha$$

which are each a topology as noted before thus by closure property of arbitrary unions we get (for every $\alpha \in A$, that is.)

$$\bigcup_{\beta \in B} U_\beta \in \tau_\alpha,$$

which gives us

$$\bigcup_{\beta \in B} U_\beta \in \bigcap_{\alpha \in A} \tau_\alpha,$$

as needed making $(X, \bigcap_{\alpha \in A} \tau_\alpha)$ a topological space. \square

(b) Given an example to show $\bigcup_{\alpha \in A} \tau_\alpha$ is not necessarily a topology given τ_α is a topology for each $\alpha \in A$.

Proof. Let our set X be given as the following three point set

$$X = \{a, b, c\}.$$

Consider the following two topologies on X ,

$$\tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{b\}\}.$$

Then their union is given by

$$\tau_1 \cup \tau_2 = \{\emptyset, X, \{a\}, \{b\}\}$$

Which is not closed under even finite unions as

$$\{a\} \cup \{b\} \notin \tau_1 \cup \tau_2$$

Thus unions of topologies need not be a topology. □

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18. Intervals in \mathbb{R} are connected

Prove intervals in \mathbb{R} are connected.

Proof. Let $I \subset \mathbb{R}$ be an interval. To show I is connected, we assume towards a contradiction that I is disconnected. That is, there is a separation of the interval

$$I = A \cup B$$

Where $A, B \in \tau_I$ (the topology on I when given the subspace topology inherited from $\mathbb{R}_{\text{standard}}$), non-empty and disjoint. Thus we are guaranteed existence of $a \in A$ and $b \in B$ such that

$$a \notin B, b \notin A.$$

let $I_0 = [a, b]$. Note that $I_0 \subseteq I$. Then we can define

$$A_0 = A \cap I_0, B_0 = B \cap I_0.$$

Then $A_0 \cup B_0$ forms a separation of I_0 . To see this we already know they are non-empty by the existence of a, b . If they were not disjoint then there exists some $\alpha \in A \cap I_0 \cap B$ contradicting A, B being disjoint as they form a separation of I . Lastly since $A, B \in \tau_I$ we have that (in the subspace topology) $A \cap I_0, B \cap I_0 \in \tau_{I_0}$. Are both open. Note that $A_0 \subset \mathbb{R}$ is non-empty thus it inherits the least upper bound property so we can define

$$c := \sup(A_0).$$

However A_0 is closed because B_0 is open thus $c \in \overline{A_0}$. and so $c \notin B_0$. As c is the supremum of A_0 , for any $x \in I_0$ with $c < x$ we have that $x \notin A_0$ thus we get

$$(x, b] \subset B_0.$$

But then c (Keep in mind that $c \in I_0$) becomes a limit point of B_0 forcing $c \in \overline{B_0}$ and since $A_0 \cup B_0$ form a separation of I_0 , $c \notin A_0$ contradicting

$$c \in I_0 = A_0 \cup B_0.$$

thus I is connected as there is no separation. □

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19. Subspace of separable is separable

If X is a separable metric space, then so is any subspace Y .

Proof. Let (X, τ) be a topological space. Assume that X is separable, then we show for any subspace $Y \subseteq X$ that Y is separable. If $Y = X$ or \emptyset we are done so we suppose $\emptyset \neq Y \subsetneq X$. We must construct a countable dense subset for Y . Let $A \subset X$ be a countable dense subset. Then we can write

$$A = \{a_1, a_2, \dots\}$$

such that

$$\overline{A} = X.$$

That is, for every $x \in X$ and $U_x \in \tau$ containing x we have

$$U_x \setminus \{x\} \cap A \neq \emptyset.$$

If we take $y \in Y$, then for any given $\varepsilon > 0$ we have that

$$B_\varepsilon(y) \cap A \neq \emptyset.$$

As the intersection is non-empty let $x_k \in B_\varepsilon(y) \cap A$. I.e.,

$$x_k \in \{B_\varepsilon(y) \cap A : \varepsilon \in \mathbb{R}^+\}.$$

Thus

$$B_\varepsilon(x_k) \cap Y \neq \emptyset.$$

Then we can take

$$\mathcal{B} = \{(k, \varepsilon) : B_\varepsilon(x_k) \cap Y \neq \emptyset\},$$

which is non-empty. So for each (k, ε) take $y_{k,\varepsilon} \in B_\varepsilon(x_k) \cap Y \neq \emptyset$ and let

$$Z = \{y_{k,\varepsilon} : (k, \varepsilon) \in \mathcal{B}\}.$$

And so we have that $Z \subset Y$ is countable since the elements are pulled from elements who are in A which is countable. We must show Z is dense in Y . That is, we must show

$$\overline{Z} = Y.$$

Let $y \in Y$ and $r > 0$ and choose ε such that

$$\varepsilon \leq \frac{r}{2}.$$

Then we can always find a $k \in \mathbb{N}$ such that

$$x_k \in B_\varepsilon(y).$$

Then $(k, \varepsilon) \in \mathcal{B}$ and by the triangle inequality,

$$\begin{aligned} d(y, y_{k,\varepsilon}) &\leq d(y, x_k) + d(x_k, y_{k,\varepsilon}) \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &\leq r. \end{aligned}$$

Thus $y_{k,\varepsilon} \in B_r(y)$ and thus $y \in \overline{Z}$ making Z dense in Y so Y is separable. □

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20. Hausdorff Diagonal

Let (X, τ) be a topological space. Then X is Hausdorff if and only if $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proof. First let us assume that Δ is closed in $X \times X$. We wish to show X is Hausdorff so let $x, y \in X$ be arbitrary. As $\Delta \subset X \times X$ is closed, by definition we know that $\Delta^c \in \tau$. Then there exists a basis element of the form

$$U \times V \quad ; U, V \in \tau,$$

such that

$$(x, y) \in U \times V \subset \Delta^c.$$

And so we have that

$$(U \times V) \cap \Delta = \emptyset$$

which gives us $x \in U, y \in V$. Lastly, I claim that

$$U \cap V = \emptyset.$$

If not, then there exists some $z \in U \cap V$ forcing

$$(z, z) \in U \times V.$$

Moreover, $(z, z) \in \Delta$ by definition, contradicting disjointness of $U \times V$ and Δ and so X is Hausdorff. On the other hand, let us assume that X is Hausdorff and we wish to show that $\Delta \subset X \times X$ is closed. We show this by showing Δ^c is open. Let $x, y \in \Delta^c$. As X is Hausdorff we are guaranteed the existence of $U_x, U_y \in \tau$ such that

$$U_x \cap U_y = \emptyset.$$

I claim that

$$(U_x \times U_y) \cap \Delta.$$

Let $(a, b) \in (U_x \times U_y) \cap \Delta$ then $a = b \in U_x \cap U_y$ a contradiction and thus $\Delta^c \in \tau$ and so $\Delta \subset X \times X$ is closed. \square

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21. Equivalence of continuity in metric space

A function f is continuous using open sets if and only if it is continuous in the $\varepsilon - \delta$ sense. Let τ_X, τ_Y denote topologies in X and Y respectively.

Proof. Let X, Y be metric spaces and

$$f : X \rightarrow Y$$

a map between them. First we will assume f is open set continuous. That is, if $V \in \tau_Y$ then $f^{-1}(V) \in \tau_X$. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then $f(x_0) \in Y$ and we have that

$$(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \in \tau_Y.$$

Then since f is continuous we get

$$x_0 \in f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)) \in \tau_X.$$

So we can find a basis element containing x_0 fully contained in $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$. I.e., there exists $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)).$$

But then we have

$$f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

Thus for any $\varepsilon > 0$ we can always find a $\delta > 0$ such that if

$$|x - x_0| < \delta$$

then

$$|f(x) - f(x_0)| < \varepsilon$$

making f continuous in the $\varepsilon - \delta$ senses.

On the other hand suppose that f is continuous in the $\varepsilon - \delta$ sense and let

$$f : X \rightarrow Y$$

be our map. Let $V \in \tau_Y$. We wish to show $f^{-1}(V) \in \tau_X$. Let $x_0 \in f^{-1}(V)$, then $f(x_0) \in V \in \tau_Y$ and so there exists an $\varepsilon > 0$ such that

$$f(x_0) - \varepsilon, f(x_0) + \varepsilon \subset V.$$

And since f is continuous in $\varepsilon - \delta$ sense are guaranteed the existence of some $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}(V).$$

This δ ball is the neighborhood of x_0 properly contained in $f^{-1}(V)$ thus $f^{-1}(V) \in \tau_X$ as needed. \square

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22. Continuous function over compact space

(a) Show continuous over compact attain a max.

Proof. Let $f : X \rightarrow \mathbb{R}$ be continuous. Suppose that X is compact. Since compactness is a topological property, $f(X)$ is compact in \mathbb{R} . By Heine-Borel, subsets of \mathbb{R} under the standard topology are compact if and only if they are both closed and bounded. Thus $f(X) \subset \mathbb{R}$ is closed and bounded. Since $f(X)$ is bounded there exists some $M \in \mathbb{R}$ such that for every $x \in X$ we have that

$$|f(x)| \leq M.$$

This tells us the supremum not only exists but is finite thus we can define

$$a := \sup_{x \in X} f(x)$$

Making a a limit point of $f(X)$ which is closed in Y forcing $a \in f(X)$ then by definition we have that

$$f(x) \leq a$$

for every $f(x) \in f(X)$. Hence $f(X)$ attains its max. □

(b) Show continuous over compact is uniform.

Proof. Let

$$f : X \rightarrow Y$$

be a map between metric spaces. If f is continuous and X is compact, prove that f is uniformly continuous. I.e., δ is not dependant on each point. Since f is continuous, for each $x \in X$ and any given $\varepsilon > 0$, there is a δ_x such that if

$$d_X(x, y) < \delta_x,$$

then

$$d_Y(f(x), f(y)) < \varepsilon.$$

In other words

$$f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x)). \quad (*)$$

We now have that $\{B_{\frac{\delta_x}{2}}(x)\}_{x \in X}$ is an open cover for X . As X is compact, we can find a finite subset $A \subset X$ such that

$$X \subseteq \bigcup_{x \in A} B_{\frac{\delta_x}{2}}(x).$$

Then we can take our δ to be

$$\delta = \min_{x \in A} \left(\frac{\delta_x}{2} \right)$$

Then we have that $d_X(x, y) < \delta$, then since $x \in B_{\frac{\delta_x}{2}}(x)$ we have that $y \in B_{\delta_x}(x)$ (keep in mind $x \in A$). Lastly, if $d_X(x, y) < \delta$ I claim $d_Y(f(x), f(y)) < \varepsilon$. Applying (*) we get that

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(z)) + d_Y(f(z), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

as needed. □

Alternate proof:

Proof. Let

$$f : X \rightarrow Y$$

be a map between metric spaces. If f is continuous and X is compact, then we show f is uniformly continuous. We say f is continuous in the $\varepsilon - \delta$ at $x \in X$ if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that if

$$d_X(x, y) < \delta,$$

then

$$d_Y(f(x), f(y)) < \varepsilon.$$

I claim that if f is $\frac{\varepsilon}{2} - \delta$ continuous, then f is $\varepsilon - \frac{\delta}{2}$ continuous. To see this, note that for every $x' \in B_{\frac{\delta}{2}}(x)$ and $y \in B_{\frac{\delta}{2}}(x')$ we have that $x', y \in B_{\delta}(x)$. Thus we can compute

$$\begin{aligned} d_Y(f(x'), f(y)) &\leq d_Y(f(x'), f(x)) + d_Y(f(x), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

And we have proven the claim. Let $\varepsilon > 0$ and $x \in X$. By $\varepsilon - \delta$ continuity, there is some $n \in \mathbb{N}$ such that f is $\frac{\varepsilon}{2} - \frac{1}{n}$ continuous. Then by the claim, f is $\varepsilon - \frac{1}{2n}$ continuous. Moreover, as X is compact, it can be covered by a finite number of these balls so let n_0 be the max n value in the finite collection, then f is $\varepsilon - \frac{1}{2n_0}$ continuous on every neighborhood of X and thus on all of X . \square

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23. Countable product of separable

Prove the countable product of separable is separable.

Proof. Let $\{X_n\}_{n \in \mathbb{N}}$ be a collection of separable metric spaces. Let D_n be the associated countable dense subset and fix $x_n \in D_n$. Then for each $m \in \mathbb{N}$, we can define

$$\begin{aligned} E_m &= \{y \in D_n : y_n = x_n; \forall n \geq m\} \\ &= \prod_{1 \leq n < m} D_n \times \prod_{n \geq m} \{x_n\}. \end{aligned}$$

Which is clearly countable and thus $\bigcup_m E_m$ is countable as well. I claim that $\bigcup_m E_m$ is dense in $\prod_n X_n$. Note by the definition of product topology we can find a basis element of the form

$$B = \prod_{1 \leq n < m} V_n \times \prod_{n \geq m} X_n.$$

Where $V_n \subset X_n$ open. This is since for all but finitely many, the open sets are the whole space, in the product topology and thus

$$B \cap \bigcup_m E_m \neq \emptyset$$

forcing $\bigcup_m E_m$ to be dense in $\prod_n X_n$ as needed. □

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24. Closed intervals are compact

Prove for $a < b \in \mathbb{R}$ that $[a, b]$ is compact.

Proof. Let $[a, b]$ for $a < b$ real numbers be an interval. We would like to show it is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for $[a, b]$ (A some arbitrary indexing set). That is,

$$[a, b] \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Let

$$F := \{x \in [a, b] \mid \exists B \subset A, \text{ finite } \ni [a, x] \subseteq \bigcup_{\alpha \in B} U_\alpha\}.$$

Note that $F \neq \emptyset$. This is because $a \in F$ as the empty set is compact. We have that F is a non-empty subset of \mathbb{R} thus it inherits the least-upper-bound property. So we define

$$c := \sup F \in [a, b].$$

I claim that $c = b$. We know $c > a$ because for $\varepsilon > 0$ we there is a neighborhood U_i such that

$$[a, a + \varepsilon] \subset U_i$$

Thus $x \geq a + \varepsilon$ and we know

$$a < c < b.$$

Now take $\beta \in A$ such that $c \in U_\beta$ and choose $\varepsilon > 0$ such that

$$a \leq c - \varepsilon < c < c + \varepsilon \leq b$$

and

$$[c - \varepsilon, c + \varepsilon] \subset U_\beta$$

Since $c - \varepsilon$ is not an upper bound of F there is some c_0 with

$$c - \varepsilon \leq c_0 \leq c$$

such that $c_0 \in F$. which means $[a, c_0]$ has a finite sub-cover from our original cover. I.e.,

$$[a, c_0] \subset \bigcup_{\alpha \in B} U_\alpha$$

which implies

$$[a, c + \varepsilon] \subset \bigcup_{\alpha \in B} U_\alpha \cup U_\beta$$

Forcing $c + \varepsilon \in F$ contradicting the fact that c is the upper bound as $c < c + \varepsilon$. Thus $c = \sup F = b$. Lastly, we show $b \in F$. To see this, note for any $\varepsilon > 0$ we know that there exists some $\gamma \in A$ such that

$$[b - \varepsilon, b] \subset U_\gamma.$$

This gives us the existence of some $c_0 \in [b - \varepsilon, b]$ such that $c_0 \in F$. Then we can write

$$\begin{aligned} [a, b] &= [a, c_0] \cup [b - \varepsilon, b] \\ &\subseteq \bigcup_{\alpha \in B} U_\alpha \cup U_\gamma. \end{aligned}$$

Thus $b \in F$ and we have found a finite sub-cover for $[a, b]$ as needed. □

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25. Successor Topology

Let $X = \mathbb{N}$ and equip it with the following topology

$$\tau = \{U \subset \mathbb{N} : (2n - 1) \in U \Rightarrow 2n \in U\}.$$

Prove that (X, τ) is locally compact but not compact.

Proof. For locally compact, let $x \in \mathbb{N}$. Then x is even or odd. If x is even, then take U to be

$$U := \{x, x + 1, x + 2\}$$

and take our compact superset of U to be

$$K := \{x, x + 1, x + 2, x + 3\}$$

Then U is open and we have

$$x \in U \subset K$$

and K is compact as it is a finite point set. Next suppose x is odd, then take U to be

$$U := \{x, x + 1\}$$

and our compact set K to be

$$K := \{x, x + 1, x + 2\}$$

Then we have that

$$x \in U \subset K$$

and (X, τ) is locally compact.

For showing it is not compact, consider the open covering

$$\bigcup_{n \in \mathbb{N}} \{2n - 1, 2n\}$$

I claim this has no finite subcovering. If there was, say there existed some finite subset of \mathbb{N} call it A such that

$$\mathcal{A} = \bigcup_{n \in A} \{2n - 1, 2n\} \supset \mathbb{N}$$

Then there exists an $N \in \mathbb{N}$ such that

$$N = \max_{n \in A} 2n$$

And

$$\{2N - 1, 2N\} \notin \mathcal{A}$$

but

$$2N - 1, 2N \in \mathbb{N}.$$

Thus (X, τ) is not compact as needed. □

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26. Prime induced metric on \mathbb{Z} .

Let p be an odd prime. And define $d : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ as $d(m, n) = 0$ if $m = n$ and

$$d(m, n) = \frac{1}{r+1},$$

where r is the largest positive integer such that

$$p^r \mid m - n$$

and we write

$$p^r \parallel m - n.$$

Show d induces a metric on \mathbb{Z} .

Proof. Clearly we have $d(m, n) = 0$ by definition. Furthermore, as $r \in \mathbb{Z}^+$, we have that

$$d(m, n) = \frac{1}{r+1} \in \mathbb{R}^+$$

if $m \neq n$. For symmetry, note that

$$d(m, n) = d(n, m)$$

since

$$p^r \parallel m - n. \Rightarrow p^r \parallel n - m.$$

For the triangle inequality we would like to show

$$d(m, n) \leq d(m, l) + d(l, n)$$

Where $d(m, n) = \frac{1}{r+1}$, $d(m, l) = \frac{1}{q+1}$ where q is such that $p^q \parallel m - l$, and lastly $d(l, n) = \frac{1}{s+1}$ where s is such that $p^s \parallel l - n$. I claim that

$$\frac{1}{r+1} \leq \frac{1}{q+1} + \frac{1}{s+1}.$$

It is clear to see that since $p^q \parallel m - l$ and $p^s \parallel l - n$ then

$$p^{\min\{q, s\}} \mid (m - l) + (l - n) = m - n.$$

This implies that $r \geq \min\{q, s\}$ as $p^r \parallel m - n$. Thus we have

$$\begin{aligned} \frac{1}{r+1} &\leq \frac{1}{\min\{q, s\} + 1} \\ &\leq \frac{1}{q+1} + \frac{1}{s+1} \end{aligned}$$

and the inequality is proven, as claimed. We must next determine if the set

$$2\mathbb{Z} := \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$

is closed in \mathbb{Z} given the induced metric above. I claim $2\mathbb{Z}$ is not closed. Consider the following limit

$$\lim_{n \rightarrow \infty} 1 + p^n.$$

Since $d(1 + p^n, 1)$ goes to 0 as $n \rightarrow \infty$ we have that the limit is equal to 1, but $1 + p^n \in 2\mathbb{Z}$ thus $2\mathbb{Z}$ does not contain all of its limit points as $1 \notin 2\mathbb{Z}$. \square

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27. Compact subset of Hausdorff separated by x in its complement.

Let X be a Hausdorff topological space and $A \subset X$ compact. Show for every $x \in X \setminus A$ there exists disjoint open neighborhoods

$$U \ni x, V \supset A.$$

Proof. Let $x \in X \setminus A$. Pick an arbitrary $a \in A$. Then by the Hausdorff property, there exists open sets $U_a, V_a \subset X$ such that

$$x \in U_a, a \in V_a$$

with

$$U_a \cap V_a = \emptyset.$$

It is clear to see that

$$A \subset \bigcup_{a \in A} V_a$$

is an open cover. By compactness of A , there exists a finite subset $B \subset A$ such that

$$A \subset \bigcup_{a \in B} V_a = V.$$

Then we can take

$$U = \bigcap_{a \in B} U_a,$$

where we clearly have that $x \in U$ and since for each a , $U_a \cap V_a = \emptyset$, we have

$$U \cap V = \emptyset$$

as needed. Note $A \subset X$ is closed so we have shown X is regular. □

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28. No continuous map between \mathbb{D}^1 and S^1 .

Show there does not exist a continuous map between \mathbb{D}^1 and S^1 such that $f(z) = z$ for every $z \in S^1$.

Proof. Suppose towards a contradiction that there exists such a map. Call it f where

$$f : \mathbb{D}^1 \rightarrow S^1$$

is continuous. As \mathbb{D}^1 is convex, it is contractible. To see this fix $z_0 \in \mathbb{D}^1$ and let $z \in \mathbb{D}^1$ be arbitrary then the path containing, then define

$$\gamma : [0, 1] \times \mathbb{D}^1 \rightarrow \mathbb{D}^1$$

via

$$(t, z) \mapsto z_0 + t(z - z_0)$$

and therefore any and closed path

$$\bar{\gamma} : S^1 \rightarrow \mathbb{C}$$

can be shrunk to z_0 by composing it with γ and thus \mathbb{D}^1 contractible. For path-connected, let $a, b \in \mathbb{D}^1$ and define

$$\gamma := (1 - t)a + tb$$

for $t \in [0, 1]$. I claim $|\gamma(t)| \leq 1$.

$$\begin{aligned} |\gamma(t)| &= |(1 - t)a + tb| \\ &\leq |(1 - t)a| + |tb| \\ &= (1 - t)|a| + t|a| \\ &\leq 1 \end{aligned}$$

The last line is because $a, b \in \mathbb{D}^1$. Additionally, since

$$\pi_1(S^1) = \mathbb{Z}$$

this implies S^1 is not simply connected, contradicting the existence of such an f , thus not continuous map exists between \mathbb{D}^1 and S^1 . \square

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29. Locally compact and Hausdorff implies regular

Let X be a locally compact topological space. Prove X is regular.

Proof. Let X be a locally compact Hausdorff topological space. We show X is regular. Let $x \in X$ and $Y \subset X$ closed be given. Take K to be the compact neighborhood of x guaranteed to us by locally compactness. As $Y \cap K$ is compact and $x \notin Y \cap K$, there exists disjoint open set U, V such that

$$x \in U, Y \cap K \subset V.$$

Note that $x \in \text{int}K$ and K is closed since it is a compact subset of a Hausdorff space. Define the open set around x to be

$$\mathcal{U} = U \cap \text{int}K$$

Then $x \in \mathcal{U}$ and define

$$\mathcal{V} = V \cup (X \setminus K)$$

then we have that $F \subset \mathcal{V}$ and

$$\mathcal{U} \cap \mathcal{V} = \emptyset,$$

as needed. □

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30. Show S^1 is connected and compact.

Proof. Show S^1 is connected and compact. For compact we will show it is closed and bounded. First note $\{1\} \subset \mathbb{R}$ is closed in the map

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

defined via

$$(x, y) \mapsto x^2 + y^2.$$

Thus the preimage of $\{1\}$ under f which is continuous is closed and is precisely the unit circle, i.e.,

$$f^{-1}(\{1\}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is closed. It is bounded cause it is contained within $B_2(0)$ and thus compact. For connected, we prove it is path-connected and thus connected. To show path-connected, let $a, b \in S^1$ be distinct, then one can write

$$a = (\cos x, \sin x), b = (\cos y, \sin y).$$

Then define

$$g : [0, 1] \rightarrow S^1$$

via

$$g(t) := (\cos t(y - x) + x, \sin t(y - x) + x)$$

then clearly we have

$$g(0) = a, g(1) = b$$

and the space S^1 is path-connected and therefore connected. □

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31. Disjoint compact subspaces of Hausdorff space can be separated.

Let $A, B \subset X$ be disjoint compact spaces and suppose X is Hausdorff.

Proof. We know since X is Hausdorff and B is compact, for each $a \in A$ there exists open neighborhoods $U_a, V_a \subset X$ such that

$$a \in U_a, B \subset V_a.$$

Clearly $\{U_a\}_{a \in A}$ is a cover for A which is compact, thus there exists some finite $F \subset A$ such that

$$A \subset \bigcup_{a \in F} U_a = U$$

then take V to be

$$V = \bigcap_{a \in F} V_a,$$

then clearly we have $B \subset V$ and

$$U \cap V = \emptyset,$$

as needed. □

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32. Compute pullback of the 2-form

Let

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

be defined via

$$(u, v) \mapsto (u, v, u^2 - v^2).$$

Let ω be the 2-form

$$\omega := ydx \wedge dz + ydy \wedge dz.$$

Compute the pullback

$$F^{-1}\omega$$

. We have

$$F^{-1}\omega := F^{-1}(ydx \wedge dz + xdy \wedge dz)$$

$$:= vdu \wedge d(u^2 - v^2) + u dv \wedge d(u^2 - v^2)$$

$$= vdu \wedge (2udu - 2v dv) + u dv \wedge (2udu - 2v dv)$$

differentiating

$$= 2uv(du \wedge du) - 2v^2(du \wedge dv) + 2u^2(dv \wedge du) - 2uv(dv \wedge dv)$$

computing wedge product

$$= -2v^2(du \wedge dv) + 2u^2(dv \wedge du)$$

As $du \wedge du = 0$ same for v

$$= -2v^2(du \wedge dv) - 2u^2(du \wedge dv)$$

As $(du \wedge dv) = -(dv \wedge du)$

$$= -2(u^2 + v^2)(du \wedge dv)$$

simplifying

as needed for the pullback.

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33. Compact hausdorff spaces need be normal

Show if X is a compact Hausdorff space, then X is normal.

Proof. Let X be compact Hausdorff, let $A, B \subset X$ be disjoint and closed. Then they are both compact. And by compactness of B , and Exercise 27, for each $a \in A$, as $a \in X \setminus B$ there exists disjoint open sets

$$U_a \ni a, V_a \supset B$$

such that

$$U_a \cap V_a = \emptyset.$$

Then clearly $\{U_a\}_{a \in A}$ is an open cover for A and since A is closed subset of compact, by Exercise 2a A is compact and so there exists some finite subset $F \subset A$ such that

$$A \subset \bigcup_{a \in F} U_a = U$$

which is open and define the disjoint open set

$$B \subset \bigcap_{a \in F} V_a$$

and since $U \cap V = \emptyset$ we are done. □

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34. Closed subspace of normal space need be normal

Let X be a normal topological space. Let $A \subset X$ be closed. Show A is normal.

Proof. Let $C, D \subset A$ be closed sets. Then they are closed in X and by normalness of X we have disjoint open sets

$$U \supset C, V \supset D$$

such that

$$U \cap V = \emptyset.$$

Then take the open sets in A to be $U \cap A$ and $V \cap A$. Then clearly

$$C \subset U \cap A, D \subset V \cap A$$

as needed. □

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35. Continuous function on compact spaced need be uniform

Let

$$f : (X, d_X) \rightarrow (Y, d_Y)$$

be a continuous map defined on a compact metric space (X, d_X) . Prove f is uniform.

Proof. Let

$$f : (X, d_X) \rightarrow (Y, d_Y)$$

be give and be continuous with X compact. We show for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Given $\varepsilon > 0$, for each $x \in X$, there exists a δ_x such that

$$f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x)).$$

Now we have that $\{B_{\frac{\delta_x}{2}}(x)\}_{x \in X}$ is an open cover for X which is compact thus there exists a finite subset $A \subset X$ such that

$$X \subset \bigcup_{x \in A} B_{\frac{\delta_x}{2}}(x).$$

Since $x_1 \in B_{\frac{\delta_{x_0}}{2}}(x_0)$ for some $x_0 \in A$, we have that $x_2 \in B_{\delta_{x_0}}(x_0)$, this is by the triangle inequality:

$$\begin{aligned} d_X(x_2, x_0) &\leq d_X(x_2, x_1) + d_X(x_1, x_0) \\ &\leq \frac{\delta_{x_0}}{2} + \frac{\delta_{x_0}}{2} \\ &= \delta_{x_0} \end{aligned}$$

Then if $d_X(x_1, x_2) < \delta$, we have that

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq d_Y(f(x_1), f(x_0)) + d_Y(f(x_0), f(x_2)) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and we have the desired result, as needed. □

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36. Homotopy preserved under composition

Let $F_0, F_1 : X \rightarrow Y$ and $G_0, G_1 : Y \rightarrow Z$ be continuous such that $F_0 \simeq F_1, G_0 \simeq G_1$. Prove then that $G_0 \circ F_0 \simeq G_1 \circ F_1$.

Proof. Since $F_0 \simeq F_1$, one has that there exists a continuous map

$$H_F : X \times [0, 1] \rightarrow Y$$

such that

$$H_F(x, t) := \begin{cases} F_0(x) & t \in [0, \frac{1}{2}] \\ F_1(x) & t \in (\frac{1}{2}, 1]. \end{cases}$$

Since $G_0 \simeq G_1$, there exists a continuous map

$$H_G : Y \rightarrow Z$$

such that

$$H_G(x, t) := \begin{cases} G_0(x) & t \in [0, \frac{1}{2}] \\ G_1(x) & t \in (\frac{1}{2}, 1]. \end{cases}$$

To see that $G_0 \circ F_0 \simeq G_1 \circ F_1$ we define the map

$$H : X \times [0, 1] \rightarrow Z$$

defined via

$$H(x, t) = \begin{cases} G_0(H_F(x, 2t)) & t \in [0, \frac{1}{2}] \\ H_G(F_1(x), 2t - 1) & t \in (\frac{1}{2}, 1] \end{cases}$$

Then we have that

$$H(x, 0) = G_0(F_0(x)), H(x, 1) = G_1(F_1(x)).$$

and we have the desired homotopy. □

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37. Pullback of 1-form

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined via

$$(s, t) \mapsto (\sin t, st^2, s^2 - 1)$$

Let ω be the 1-form

$$\omega = dx + xdy + y^2dz$$

Compute the pullback $f^*\omega$.

Proof. I computed $f^*\omega$ to be (the pull back under ω) as:

$$\begin{aligned} f^*\omega &= f^*(dx + xdy + y^2dz) \\ &= d(\sin t) + \sin t d(st^2) + s^2t^4 d(s^2 - 1) \\ &= \cos t dt + \sin t(t^2 ds + 2st dt) + 2s^3t^4 ds \\ &= \cos t dt + t^2 \sin t ds + 2st \sin t dt + 2s^3t^4 ds \\ &= (\cos t + 2st \sin t) dt + (t^2 \sin t + 2s^3t^4) ds \end{aligned}$$

□

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38. Pullback of a 2-form

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

be given the 2-form $\omega = x dy \wedge dz$ compute the pull back if f is defined via

$$(\theta, \phi) \mapsto ((2 + \cos \phi) \cos \theta, (2 + \cos \phi) \sin \theta, \sin \phi)$$

.

Proof. Then we get

$$\begin{aligned} f^* \omega &= (2 + \cos \phi) \cos \theta d((2 + \cos \phi) \sin \theta) \wedge d(\sin \phi) && \text{definition} \\ &= (2 + \cos \phi) \cos \theta (-\sin \phi \sin \theta d\phi + (2 + \cos \phi) \cos \theta d\theta) \wedge \cos \phi d\phi && \text{differentiating} \\ &= (2 + \cos \phi)^2 \cos^2 \theta \cos \phi d\theta \wedge d\phi && \text{as } d\phi \wedge d\phi = 0 \end{aligned}$$

□

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39. Nested sequence of nonempty connected compact subsets of Hausdorff space has nonempty compact, connected intersection.

Let $X_1 \supset X_2 \supset X_3 \dots$ be a sequence of nonempty connected compact subspaces of \mathbb{R}^n . Prove that

$$X = \bigcap_{i=1}^{\infty} X_i$$

is nonempty, compact, and connected.

Proof. For non empty, suppose towards a contradiction that

$$X = \bigcap_{n=1}^{\infty} X_n = \emptyset.$$

Then define

$$U_n := X_1 \setminus X_n$$

which is open as the X_n are compact subsets of Hausdorff space and thus closed. Then

$$\bigcup_{n=1}^{\infty} U_n = X_1 \setminus \bigcap_{n=1}^{\infty} X_n$$

but $\bigcap_{n=1}^{\infty} X_n$ is empty thus

$$\bigcup_{n=1}^{\infty} U_n = X_1$$

which is compact thus there exists some finite subset $A \subset \mathbb{N}$ such that

$$X_1 = \bigcup_{n \in A} U_n$$

as the U_n are increasing, (since the X_n are decreasing) we have that if $N := \max_{n \in A}$, then

$$X_1 = \bigcup_{n=1}^{\infty} U_n = U_N$$

Thus $X_1 = U_N$ which forces

$$X_N = X_1 \setminus U_N = \emptyset$$

contradicting nonemptyness of all the X_n , $n \in \mathbb{N}$. For connected, suppose towards a contradiction that

$$\bigcap_{i=1}^{\infty} X_i = A \cup B$$

Where A, B are nonempty, disjoint, and their union is all of X . Let U, V be disjoint open sets containing A, B , respectively this is by separation of compact subspaces of a Hausdorff space, and define

$$F_i := X_i \setminus (U \cup V)$$

it is clear to see that as the $F_i \subset X_i$, then we have that

$$\begin{aligned}\bigcap_{i=1}^{\infty} F_i &= \bigcap_{i=1}^{\infty} X_i \setminus (U \cup V) \\ &= \emptyset \setminus (U \cup V)\end{aligned}$$

Forcing $F_k = \emptyset$ for some $k \in \{1, 2, \dots\}$, this implies, for this k , we have that

$$X_k \subset U \cup V$$

but as X_k is connected, this implies by the connected Lemma that WLOG $X_k \subset U$ forcing $V = \emptyset$ thus $B = \emptyset$ and there is no separation of the space forcing X to be connected. Furthermore, $\bigcup_{i=1}^{\infty} X_i$ is connected. Suppose

$$\bigcup_{i=1}^{\infty} X_i = C \cup D$$

where C, D are open nonempty and disjoint. Let $p \in \bigcap_{i=1}^{\infty} X_i$, then $p \in X_i$ for all i . This forces $p \in C$ or $p \in D$, WLOG let us assume $p \in C$. Then as X_i is connected, it must lie entirely within C or D but $p \in X_i$ thus $X_i \subset C$ for every i thus $D = \emptyset$ and there is no separation of the space, forcing $\bigcup_{i=1}^{\infty} X_i$ to be connected. \square

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40. Integral of form

Let θ, γ be 3-forms on \mathbb{S}^7 . Show that

$$\int_{\mathbb{S}^7} \theta \wedge d\gamma = \int_{\mathbb{S}^7} d\theta \wedge \gamma.$$

Proof. We begin by subtracting the LHS from the RHS to show

$$\int_{\mathbb{S}^7} d\theta \wedge \gamma + (-1)\theta \wedge d\gamma = 0.$$

However, we have that

$$\begin{aligned} \int_{\mathbb{S}^7} d\theta \wedge \gamma + (-1)\theta \wedge d\gamma &= \int_{\mathbb{S}^7} d(\theta \wedge \gamma) && \text{definition of wedge product on forms} \\ &= \int_{\partial\mathbb{S}^7} \theta \wedge \gamma && \text{By Stoke's Theorem} \\ &= 0 && \text{as } \partial\mathbb{S}^7 = \emptyset. \end{aligned}$$

Thus we have that

$$\int_{\mathbb{S}^7} \theta \wedge d\gamma = \int_{\mathbb{S}^7} d\theta \wedge \gamma,$$

as needed. □

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41. Quotient space onto connected space with that the fibers are connected, then X need be connected.

Let

$$f : X \rightarrow Y$$

be a quotient map. Suppose Y is connected and for each $y \in Y$,

$$f^{-1}(\{y\})$$

is connected. Show X is connected.

Proof. Suppose towards a contradiction that

$$X = A \cup B$$

where $A, B \subset X$ are both open, nonempty, and disjoint. Then $f(A), f(B)$ are open in Y , but by connectedness of Y , these sets cannot be disjoint. So there exists a y such that

$$y \in f(A) \cap f(B).$$

Then

$$f^{-1}(\{y\}) = (A \cap f^{-1}(\{y\})) \cup (B \cap f^{-1}(\{y\}))$$

where both sets are open in the subspace topology and both nonempty. As $y \in f(A)$ implies $f(x) = y$ for some $x \in A$. Same for B . By connectivity of $f^{-1}(\{y\})$ these sets cannot be disjoint, thus there exists some $x \in A \cap B$, contradicting A, B being disjoint. \square

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42. Open subspaces of compact Hausdorff spaces need be locally compact.

Let X be a compact Hausdorff space. Let $U \subset X$ be open. We show U is locally compact.

Proof. Let $x \in U$, then since U is open, there exists an open set U_0 such that

$$x \in U_0 \subset U.$$

By nature of the subspace topology, there exists an open set $V \subset X$ such that

$$U_0 = U \cap V.$$

As X is compact Hausdorff, it is normal, so there exists a smaller open set $W \subset X$ such that

$$x \in W \in \overline{W} \subset V \subset X,$$

then we have that

$$x \in W \cap U \subset \overline{W} \cap U \subset U_0 \subset U$$

□

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43. Normal iff Closed sets can be covered

Let (X, τ) be a topological space. Then X is normal if and only if for any closed set $C \subset X$ and $U \in \tau$ with

$$C \subset U,$$

there exists $V \in \tau$ such that

$$C \subset V \subset \bar{V} \subset U.$$

Proof. Suppose first that X is normal, Let $C \subset X$ be closed and let $U \in \tau$ be such that

$$C \subset U.$$

Then $X \setminus U$ is closed and thus there exists $V, W \in \tau$ with

$$C \subset V, X \setminus U \subset W$$

such that

$$V \cap W = \emptyset.$$

Then $V \subset X \setminus W$ which is closed forcing

$$\bar{V} \subset X \setminus W$$

hence

$$C \subset V \subset \bar{V} \subset X \setminus W \subset U$$

as needed.

On the other hand, suppose every closed set $C \subset X$ and every $U \in \tau$ with $C \subset U$ we can find a smaller open set $V \in \tau$ such that

$$C \subset V \subset \bar{V} \subset U$$

Let $C_1, C_2 \subset X$ be closed and disjoint. As they are disjoint, we have that

$$C_1 \subset X \setminus C_2$$

and since C_2 is closed, $X \setminus C_2$ is open and thus there exists $V \in \tau$ such that

$$C_1 \subset V \subset \bar{V} \subset X \setminus C_2.$$

Then, $X \setminus \bar{V}$ is open and contains C_2 and is disjoint from V and we have normality. \square

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44. Continuous iff every net converges

Let

$$f : X \rightarrow Y$$

be a map of topological spaces. Then show f is continuous if and only if for every net $(x_\alpha)_{\alpha \in A}$ converging to x , we have $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$.

Proof. Suppose first that f is continuous. Let $(x_\alpha)_{\alpha \in A}$ converge to x . Let $V \subset Y$ be open containing $f(x)$. Then

$$x \in f^{-1}(V)$$

which is open by continuity of f , thus there exists some $\alpha_0 \in A$ such that

$$x_\alpha \in f^{-1}(V)$$

for all $\alpha \geq \alpha_0$. Then we have

$$f(x_\alpha) \in V$$

forcing $f(x_\alpha) \rightarrow f(x)$.

Conversely, suppose that if $(x_\alpha)_{\alpha \in A}$ converges to x , then $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$. And assume f is discontinuous at x . I.e., there exists some open set $V \subset Y$ containing $f(x)$ such that

$$x \in (f^{-1}(V))^\circ = \overline{f^{-1}(V^c)}.$$

Then one can conclude that there exists a net (x_α) such that for all $\alpha \geq \alpha_0$ one has

$$x_\alpha \in f^{-1}(V^c).$$

Then for all $\alpha \geq \alpha_0$ one has that since $f(x) \in V$ that

$$f(x_\alpha) \in V.$$

This is a contradiction since $f(x_\alpha) \in V^c$ thus f is continuous. □

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45. Hausdorff under continuous function

Let

$$f : X \rightarrow Y$$

be 1-1 and continuous. If X is Hausdorff, this does not imply Y is Hausdorff. Take X and topological space with discrete topology, f the constant map and Y and infinite topological space with the cofinite topology.

On the other hand, suppose Y is Hausdorff, we show X must also be Hausdorff.

Proof. Let $x_1, x_2 \in X$ be distinct. As f is 1-1, $f(x_1) \neq f(x_2) \in Y$. As they're distinct in a Hausdorff space, there exists open sets $V_1, V_2 \subset Y$ with $x_1 \in V_1, x_2 \in V_2$ such that

$$V_1 \cap V_2 = \emptyset.$$

Then since f is continuous,

$$f^{-1}(V_1), f^{-1}(V_2) \subset X$$

are open and since f and since V_1, V_2 are disjoint, $f^{-1}(V_1), f^{-1}(V_2)$ are disjoint and contain x_1, x_2 respectively, thus X is Hausdorff. \square

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46. Hausdorff preserved under homeo

Let

$$f : X \rightarrow Y$$

be a homeomorphism. If X is Hausdorff, then so is Y .

Proof. Let $y_1, y_2 \in Y$ be distinct. By surjectivity of f , there exists $x_1, x_2 \in X$ such that

$$f(x_1) = y_1, f(x_2) = y_2.$$

Since f is a bijection, we have that

$$f(x_1) \neq f(x_2)$$

and since X is Hausdorff, there exists open sets

$$U_1, U_2 \subset X$$

with $x_1 \in U_1, x_2 \in U_2$ such that

$$U_1 \cap U_2 = \emptyset.$$

Then since f is 1-1,

$$f(U_1) \cap f(U_2) = \emptyset$$

and we have that

$$y_1 \in f(U_1), y_2 \in f(U_2)$$

which are open as f is an open map.

□

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47. Open and connected subset of \mathbb{R}^2 implies connected

Let $U \subset \mathbb{R}^2$ be open and path-connected. We prove U is path-connected.

Proof. Let $U \subset \mathbb{R}^2$ be connected and open. Let $a \in U$ and $A \subset U$ which can join a . Define

$$C = U \setminus A$$

and therefore $A \cap C = \emptyset$. We wish to prove $A \subset U$ is non-empty closed and open and thus all of U making U path connected. For non-empty, clearly $a \in A$.

For open let $x \in A$, then by openness of U , there exists an $\epsilon > 0$ such that

$$B_\epsilon(x) \subset U.$$

As open balls are convex, they are path-connected thus for any point $y \in B_\epsilon(x)$ there exists a path from x to y

$$\phi : [x, y] \rightarrow \mathbb{R}^2$$

But as A is the set of points that joins a , there is path from x to a , namely

$$\psi : [a, x] \rightarrow \mathbb{R}^2$$

Then there exists a path from y to a thus $y \in A$. And as $y \in B_\epsilon(x)$, we conclude that

$$B_\epsilon(x) \subset A$$

forcing $A \subset U$ to be open.

To show $A \subset U$ is closed we show $C = U \setminus A$ is open which is the set of points that cannot be joined with a via continuous path within U thus C is open since every $y \in B_\epsilon(x)$ cannot be joined for $\epsilon > 0$ such that

$$B_\epsilon(x) \subset U,$$

otherwise, we can join a by y and y by x thus a with x contradicting the definition of C . \square

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48. Open map of compact space into connected Hausdorff space.

Let $f : X \rightarrow Y$ be continuous and open. Suppose X is compact and Y is connected and Hausdorff. Then f is onto.

Proof. It suffices to show

$$f(X) = Y.$$

As Y is connected, it suffices to prove $f(X) \subset Y$ is both open and closed. Since X is open and f is an open map we can conclude that $f(X) \subset Y$ is open. As f is continuous and X is compact, $f(X) \subset Y$ is compact in a Hausdorff space which implies it is closed forcing

$$f(X) = Y$$

as needed. □

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49. Metric spaces are normal

Let (X, d) be a metric space. Then X is a normal space.

Proof. Let (X, d) be a metric space. Let $E, F \subset X$ be disjoint open sets. Then for each $x \in E$, there exists some $\varepsilon_x > 0$ such that

$$B_{\varepsilon_x}(x) \cap F = \emptyset.$$

Similarly, for each $y \in F$ there exists some $\varepsilon_y > 0$ such that

$$B_{\varepsilon_y}(y) \cap E = \emptyset.$$

Then I claim the disjoint open sets of E, F are given as

$$U := \bigcup_{x \in E} B_{\frac{\varepsilon_x}{2}}(x), V := \bigcup_{y \in F} B_{\frac{\varepsilon_y}{2}}(y).$$

First off, it is clear to see $E \subset U$ and $F \subset V$. Lastly, I claim

$$U \cap V = \emptyset.$$

If not, then there exists some $z \in U \cap V$. Then we have

$$d(x, z) < \frac{\varepsilon_x}{2}, d(y, z) < \frac{\varepsilon_y}{2}.$$

Then by the triangle inequality of metric spaces we have

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(y, z) \\ &< \frac{\varepsilon_x + \varepsilon_y}{2} \\ &\leq \max\{\varepsilon_x, \varepsilon_y\}. \end{aligned}$$

But this forces either

$$x \in B_{\varepsilon_y}(y)$$

or

$$y \in B_{\varepsilon_x}(x).$$

And this contradicts our choice of $\varepsilon_x, \varepsilon_y$ and therefore $U \cap V = \emptyset$ and contain E, F respectively thus (X, d) is a normal space. \square

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50. If the product is Hausdorff and normal, so is each.

(a) Let $X = \prod_{\alpha \in J} X_\alpha$ is Hausdorff, then for each $\alpha \in J$, X_α is Hausdorff. (b) If $X = \prod_{\alpha \in J} X_\alpha$ is normal then so is X_α for each $\alpha \in J$.

Proof. (a) Assume $X = \prod_{\alpha \in J} X_\alpha$ is Hausdorff. Fix some $\beta \in J$ and consider the distinct elements $x, y \in X_\beta$. Now choose $f, g \in X$ such that

$$f(\beta) = x, g(\beta) = y,$$

and

$$f(\alpha) = g(\alpha); \quad \text{for all } \alpha \neq \beta. (*)$$

As $f \neq g \in X$, by Hausdorffness, there exists open disjoint subsets of X , call them U, V with

$$f \in U, g \in V.$$

By the definition of the product topology, there exists families of open sets $\{U_\alpha\}_\alpha, \{V_\alpha\}_\alpha$ such that

$$f \in \prod_{\alpha} U_\alpha \subset U, g \in \prod_{\alpha} V_\alpha \subset V.$$

Here we have that

$$U_\alpha = X$$

for all but finitely many values of α and the same goes for the V_α . Then by (*) we have that

$$U_\beta \cap V_\beta = \emptyset.$$

Where $x \in U_\beta, y \in V_\beta$ forcing X_β to be Hausdorff.

(b) Suppose next that $X = \prod_{\alpha \in J} X_\alpha$ is normal. Fix $\beta \in J$ and consider disjoint closed sets $A, B \subset X$. Then as the projection map

$$p_\beta : X \rightarrow X_\beta$$

is continuous, we have that

$$p_\beta^{-1}(A), p_\beta^{-1}(B)$$

are closed and disjoint in X . Thus by normality of X we have that there exists disjoint open sets $U, V \subset X$ with

$$p_\beta^{-1}(A) \subset U, p_\beta^{-1}(B) \subset V.$$

Next, fix some $f_0 \in X$ and define

$$g : X_\beta \rightarrow X$$

via

$$(g(x))_\beta = \begin{cases} x & \beta = \alpha \\ f_0(\beta) & \beta \neq \alpha \end{cases}$$

Then one can see that $g(X_\beta) \cong X_\beta$. Then it is clear to see the disjoint open sets containing A, B respectively are

$$p_\beta(U \cap g(X_\beta)) = g^{-1}(U)$$

and

$$p_\beta(V \cap g(X_\beta)) = g^{-1}(V)$$

thus X_β is a normal space. □

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51. Separable metric spaces are second countable.

Let (X, d) be a separable metric space. Then X is second countable.

Proof. Let (x, d) be a separable metric space. I.e., there exists some $Y \subset X$ with $\overline{Y} = X$. Define then

$$\mathcal{B} = \{B_{\frac{1}{n}}(x) : x \in Y, n \in \mathbb{Z}^+\}.$$

Then \mathcal{B} is clearly countable. If τ is the topology on X , let

$$y \in U \in \tau.$$

I claim that $B_{\frac{1}{n}}(x) \subset U$. As $U \in \tau$, there exists some $\varepsilon > 0$ such that

$$B_\varepsilon(y) \subset U.$$

However, there exists an $n \in \mathbb{Z}^+$ such that

$$n > \frac{\varepsilon}{2}.$$

I.e., $\varepsilon > \frac{2}{n}$. That is,

$$B_{\frac{2}{n}}(y) \subset B_\varepsilon(y) \subset U.$$

By density of Y in X , there exists x with

$$x \in Y \cap B_{\frac{1}{n}}(y).$$

This implies $y \in B_{\frac{1}{n}}(x)$. Then for every $z \in B_{\frac{1}{n}}(x)$ we have by the triangle inequality that

$$\begin{aligned} d(z, y) &\leq d(z, x) + d(x, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n} \end{aligned}$$

That is,

$$B_{\frac{1}{n}}(x) \subset B_{\frac{2}{n}}(y) \subset U$$

and the claim is proven thus \mathcal{B} is a countable basis for X . □

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Counters

1. Any space with the indiscrete topology is connected. With the discrete topology everything is disconnected.
2. \mathbb{R}_l is not connected. Take $A = (-\infty, 0)$, $B = [0, \infty)$ as separation components.
3. \mathbb{R}_K is connected hausdorff, but not path connected, not compact, not regulars
4. $[0, 1]$ is no longer compact in the K -topology, K is an infinite subspace in closed unit with no limit point in $[0, 1]$.
5. In discrete topology, $\overline{\mathbb{Q}} = \mathbb{Q}$
6. $\overline{(0, 1)}$ in \mathbb{R}_l is $[0, 1)$.
7. To show $\overline{\bigcup A_i}$ is not always contained in $\bigcup \overline{A_i}$ Consider $A = \{r_i\}$ is an enumeration of the rationals, then

$$\begin{aligned}
 \overline{\bigcup A_i} &= \overline{\bigcup \{r_i\}} \\
 &= \overline{\mathbb{Q}} \\
 &= \mathbb{R} \\
 &\not\subseteq \bigcup \overline{A_i} \\
 &= \bigcup \{r_i\} \\
 &= \mathbb{Q}.
 \end{aligned}$$

as needed.

8. To show $\overline{\text{int}(A)}$ is not always contained in $\text{int}(\overline{A})$ consider $A = [0, 1)$.
9. To show $\text{int}(\overline{A})$ is not always contained in $\overline{\text{int}(A)}$ consider $A = \mathbb{Q}$.
10. indiscrete everything connected, in the discrete not.
11. To show $\overline{A \setminus B}$ is not contained in $\overline{A} \setminus \overline{B}$ take \mathbb{R} and \mathbb{Q} .
12. \mathbb{R} with cofinite topology is compact but not Hausdorff.
13. Note that the boundary of subset of a top space, $\overline{A} \setminus \text{int}(A)$ does not contain all limit points of A . Take $A = [0, 1]$ Then the boundary is $\{0, 1\}$ but the set of limit points of A is all of A .
14. $\overline{\text{int}(A)}$ does not contain all limit points of A take $A = \mathbb{Q}$.

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