

# Point-Set Topology/Geometry/Ag1 Topology Select Solutions

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## 1. Nested closed subsets of compact

Let  $(X, \tau)$  be a compact topological space. Let  $E_1 \supset E_2 \supset \dots$  be closed subsets of  $X$ . If  $X$  is compact, prove that  $\bigcap_{i \in I} E_i \neq \emptyset$ .

*Proof.* Let us assume toward a contradiction that

$$\bigcap_{n \in \mathbb{N}} E_n = \emptyset$$

Since for each  $n \in \mathbb{N}$  we know  $E_n$  is closed in  $X$  and they are decreasing, we know the  $X \setminus E_n$  are increasing. Then we have that for each  $n$ ,

$$X \subseteq \bigcup_{n \in \mathbb{N}} X \setminus E_n.$$

By compactness of  $X$  we have the existence of a finite subset  $A \subset \mathbb{N}$  such that

$$\begin{aligned} X &\subseteq \bigcup_{n \in A} X \setminus E_n \\ &= X \setminus E_M, \end{aligned}$$

Where  $M \in A$  is the maximal element in terms of set containment. But this implies

$$E_n = \emptyset$$

For every  $n$  such that  $1 \leq n < M$ , contradicting non-emptiness. □

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## 2 Compactness

(a) Closed subspaces of compact topological spaces need be compact.

*Proof.* Let  $X$  be a compact topological space. Suppose  $Y \subseteq X$  is closed. We would like to show that  $Y$  is compact. Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $Y$ . That is,

$$Y \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

As  $Y \subseteq X$  is closed, its complement in  $X$ ,  $X \setminus Y$ , is open in  $X$ . Moreover, the union of the  $U_\alpha$  together with  $X \setminus Y$  forms an open cover for  $X$ , i.e.,

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha \cup X \setminus Y.$$

Since  $X$  is compact, we know there exists a finite subset  $B \subseteq A$  such that

$$X \subseteq \bigcup_{i \in B} U_{\alpha_i} \cup X \setminus Y$$

And since  $Y \subseteq X$  we have that

$$Y \subseteq \bigcup_{i \in B} U_{\alpha_i}$$

thus we have found a finite subcover of  $\{U_\alpha\}$  that cover  $Y$  and so  $Y$  is compact as needed.  $\square$

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(b) Compact subspaces of Hausdorff topological spaces need be closed.

*Proof.* Let  $X$  be a Hausdorff topological space. Suppose  $Y \subseteq X$  is compact. We wish to show  $Y$  is closed in  $X$ . It suffices to show its complement  $X \setminus Y$  is open in  $X$ . Let  $x \in X \setminus Y$ , as  $X$  is Hausdorff, for each  $y \in Y$  there exists open sets  $U_y \subseteq X \setminus Y$ ,  $V_y \subseteq Y$  with  $x \in U_y$ ,  $y \in V_y$  such that

$$U_y \cap V_y = \emptyset.$$

Since  $V_y \subseteq Y$  is open,  $\{V_y \mid y \in Y\}$  is an open cover for  $Y$ , that is,

$$Y \subseteq \bigcup_{y \in Y} V_y.$$

By compactness of  $Y$  there exists a finite subset  $A \subseteq Y$  such that

$$Y \subseteq \bigcup_{y \in A} V_y = V.$$

Then the finite intersection  $U = \bigcap_{y \in A} U_y$  is an open neighborhood of  $x$  disjoint from  $V$ , namely

$$x \in U \subseteq X \setminus Y$$

and thus  $X \setminus Y$  is open in  $X$  forcing  $Y \subseteq X$  to be closed as needed.  $\square$

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(c) The image of a compact topological space need be compact under a continuous map.

*Proof.* Let  $f : X \rightarrow Y$  be a continuous map between topological spaces. Suppose  $X$  is compact. We would like to show that  $f(X) \subseteq Y$  is compact. Let  $\{V_\alpha\}_{\alpha \in A}$  be an arbitrary open cover for  $f(X)$ . That is,

$$f(X) \subseteq \bigcup_{\alpha \in A} V_\alpha$$

where  $V_\alpha \subseteq Y$  is open for each  $\alpha \in A$ . As  $f$  is continuous, we have that

$$f^{-1}(V_\alpha) \subseteq X$$

is open for each  $\alpha \in A$  as well. Then  $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$  is an open cover for  $X$  and since  $X$  is compact we have the existence of a finite subset  $B \subseteq A$  such that

$$X \subseteq \bigcup_{b \in B} f^{-1}(V_b)$$

then it follows that

$$f(X) \subseteq \bigcup_{b \in B} V_b$$

thus  $f(X)$  is compact. □

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### 3. Regular space equivalence

(a) Define a regular space.

*Proof.* A topological space  $X$  is called *regular* or  $T_3$  if  $\forall x \in X$  and any closed subset  $F \subset X$  not containing  $x$ ,  $\exists U, V \subset X$  open with  $x \in U, F \subset V \ni$

$$U \cap V = \emptyset.$$

Namely, we can separate points from closed sets using open sets. □

Assume  $X$  is Hausdorff. Then  $X$  is regular if and only if for every  $x \in X$  with neighborhood  $U \subseteq X$  there exists  $V \subseteq X$  open with  $x \in V$  such that  $\bar{V} \subset U$ .

*Proof.* Let  $X$  be a Hausdorff topological space. Suppose first that  $X$  is also regular. Let  $x \in X$  and let  $U_x \subseteq X$  be an open neighborhood of  $x$ . Then  $F = X \setminus U_x$  is closed by definition. Since  $X$  is a regular space we can separate  $x$  and  $F$  with open subsets of  $X$ , that is, there exists  $V, W \subseteq X$  open with  $x \in V, F \subset W$  such that

$$V \cap W = \emptyset.$$

As  $x \in U_x$  we have that

$$V \cap F = \emptyset$$

forcing  $\bar{V} \subseteq U_x$  as needed.

Next suppose for each  $x \in X$  and open neighborhood of  $x$  say  $U_x \subseteq X$  there exists an open neighborhood  $V \subseteq X$  with  $x \in V$  such that  $\bar{V} \subseteq U_x$ . Let  $x \in X \setminus F$  where  $F \subseteq X$  is closed. We wish to separate these via open subsets of  $X$ . As  $F$  is closed it follows that  $X \setminus F \subseteq X$  is open. Then by our assumption we are guaranteed the existence of an open set  $V \subseteq X$  such that

$$\bar{V} \subseteq X \setminus F.$$

Here  $x \in V$  is an open neighborhood of  $x$  and similarly  $X \setminus \bar{V}$  is an open set containing  $F$  such that

$$V \cap X \setminus \bar{V} = \emptyset$$

thus  $X$  is regular as needed. □

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## 4. Non-disjoint union of connected is connected

(a) Let  $\{X_\alpha\}_{\alpha \in I}$  be a collection of topological spaces. If for each  $\alpha \in I$   $X_\alpha$  is connected and

$$\bigcap_{\alpha \in I} X_\alpha \neq \emptyset,$$

then we have that

$$\bigcup_{\alpha \in I} X_\alpha$$

is connected as well.

*Note that this fails if we swap the conclusions union with intersection, take  $X_1 := S^1, X_2 := \{(x, y) : x = y \in \mathbb{R}\}$ , so the circle and the line, their union is connected, their intersection is two disjoint points however.*

*Proof.* First, Recall the Connected lemma:

Let  $(X, \tau)$  be a topological space. If  $A \cup B$  is a separation of the space and  $Y \subset X$  is a connected subspace, then  $Y$  lies entirely in  $A$  or  $B$ .

The proof of this is in Exercise 16. Now assume towards a contradiction that  $\bigcup_{\alpha \in I} X_\alpha$  is disconnected. That is, there is a separation. I.e.,

$$\bigcup_{\alpha \in I} X_\alpha = A \cup B$$

Where  $A, B \in \tau$  are non-empty and disjoint. Since the intersection of the  $X_\alpha$  is non-empty, let  $x \in \bigcap_{\alpha \in I} X_\alpha$ , then  $x \in A$  or in  $B$ , let us say  $x \in A$ . But then  $B$  is non-empty thus there exists some  $y \in B$ . But then  $y \in X_\beta$  for some  $\beta \in I$  and also  $x \in X_\beta$  contradicting the connected lemma as  $X_\beta$  is connected it must lie entirely in  $A$  or  $B$  and so  $\bigcup_{\alpha \in I} X_\alpha$  is connected.  $\square$

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(b) The image of a connected space under continuous function need be connected.

*Proof.* Let  $f : X \rightarrow Y$  be a continuous map of topological space. Suppose that  $X$  is connected. We wish to show that  $f(X)$  is connected as well. Let us suppose towards a contradiction that  $f(X)$  has a separation, that is,

$$f(X) = A \cup B$$

where  $A, B \subsetneq f(X)$  are both nonempty and open. Since  $f$  is continuous,  $f^{-1}(A), f^{-1}(B) \subset X$  are both open. Moreover, their union is all of  $X$  and thus we have formed a separation of  $X$  which is connected, a contradiction and so  $f(X)$  is connected.  $\square$

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(c) Let  $X, Y$  be connected. Show  $X \times Y$  is connected in the product topology.

*Proof.* Suppose  $X, Y$  are connected topological spaces. We wish to show that with respect to the product topology,  $X \times Y$  is connected as well. Fix  $(x_0, y_0) \in X \times Y$ . As  $X$  is connected and homeomorphic to the slice  $X \times \{y_0\}$ , it follows that  $X \times \{y_0\}$  is connected. Similarly, for each  $x \in X$  we have that the slice  $\{x\} \times Y$  is connected as well. We can now define

$$T_x = (X \times \{y_0\}) \cup (\{x\} \times Y)$$

Then  $\bigcup_{x \in X} T_x$  is connected by part (a) as the intersection consists of  $(x, y_0)$ . As  $\bigcup_{x \in X} T_x$  is all of  $X \times Y$ , we are done.  $\square$

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## 5. Equivalence for closed in metric space

Let  $(X, d)$  be a metric space with  $C \subset X$  and some  $p \in X$  a point. Prove  $C$  is closed if and only if  $C \cap \overline{B_R(p)}$  is closed for any  $R > 0$  where

$$\overline{B_R(p)} = \{x \in X \mid d(x, p) \leq R\}.$$

*Proof.* First suppose  $C$  is closed. Since arbitrary intersections of closed spaces need be closed we have that the intersection

$$C \cap \overline{B_R(p)}$$

is closed as  $\overline{B_R(p)}$  is closed by definition.

On the other hand suppose that for some  $p \in X$  and any  $R > 0$  the intersection

$$C \cap \overline{B_R(p)}$$

is closed. We wish to show that  $C$  is closed, i.e.,  $C = \overline{C}$ . Clearly we have that  $C \subseteq \overline{C}$  thus we are left to show that  $\overline{C} \subseteq C$ . So let  $x \in \overline{C}$  be a limit point. We must show  $x \in C$ . As  $X$  is a metric space we can put  $x$  in an epsilon ball, that is,

$$x \in B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}.$$

As  $C \cap \overline{B_R(p)}$  is closed for any  $R$ , we can take  $R = d(x, p) + \varepsilon$ . Then we have that

$$B_\varepsilon(x) \subset B_R(p) \subset \overline{B_R(p)}.$$

As  $x$  is a limit point of  $C$ ,  $B_\varepsilon(x) \setminus \{x\}$  intersects  $C$  non-trivially thus  $x$  is a limit point of  $C \cap \overline{B_R(p)}$  hence  $x \in C \cap \overline{B_R(p)}$  forcing  $x \in C$  as needed and  $C$  is closed.  $\square$

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## 6. Hausdorff & second-countable

(a) Define *second-countable*

*Proof.* A topological space  $X$  is said to be *second-countable* if it has a countable basis.  $\square$

(b) Define Hausdorff.

*Proof.* A topological space  $X$  is said to be Hausdorff if for any pair of distinct elements  $x, y \in X$  we can find open subsets of  $X$  say  $U, V$  with  $x \in U, y \in V$  such that

$$U \cap V = \emptyset.$$

$\square$

(c) Prove or disprove: Every metric space equipped with the metric topology is Hausdorff.

*Proof.* Let  $(X, d)$  be a metric space. Then  $d$  induces a topology on  $X$ , namely the collection of epsilon balls, that is,

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}.$$

This collection forms a basis for a topology on  $X$ . I claim  $X$  is Hausdorff. Let  $x, y \in X$  be distinct. Then we have that  $d(x, y) > 0$  so denote this distance by  $\varepsilon_0$ . Then we have basis elements  $x \in B_{\frac{\varepsilon_0}{2}}(x), y \in B_{\frac{\varepsilon_0}{2}}(y)$ . It suffices to show

$$B_{\frac{\varepsilon_0}{2}}(x) \cap B_{\frac{\varepsilon_0}{2}}(y) = \emptyset$$

Suppose there exists some  $z \in B_{\frac{\varepsilon_0}{2}}(x) \cup B_{\frac{\varepsilon_0}{2}}(y)$  then  $d(x, z), d(z, y) < \frac{\varepsilon_0}{2}$ . And so by the triangle inequality we have

$$\begin{aligned} \varepsilon_0 &= d(x, y) \\ &\leq d(x, z) + d(z, y) \\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} \\ &= \varepsilon_0 \end{aligned}$$

a contradiction forcing the intersection to be empty as needed thus  $(X, d)$  is Hausdorff.  $\square$

(d) Prove or disprove: Every metric space equipped with the metric topology is *second-countable*.

*Proof.* Consider  $\mathbb{R}$  as a metric space equipped with discrete metric, that is

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Here the basis elements consist of singleton sets.  $\mathbb{R}$  has an uncountable number of points, the basis (the singletons) is uncountable thus not second-countable.  $\square$

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## 7. The product topology

Consider the Product space  $Y = \prod_{n=1}^{\infty} [0, 1]$  with the product topology.

(a) Prove  $Y$  is Hausdorff.

*Proof.* As  $[0, 1]$  is Hausdorff (It is a subspace of the reals which are Hausdorff and it is easy to check subspaces of Hausdorff under subspace topology need be Hausdorff as well) we verify that given a collection  $\{(X_i, \tau_i)\}_{i \in I}$  of topological spaces, if they are all assumed to be Hausdorff, then their product

$$(\prod_{i \in I} X_i, \tau_X)$$

(for short we denote  $\prod_{i \in I} X_i$  by  $X$ ) is Hausdorff when  $\tau_X$  is the product topology. Before proceeding, small lemma:

Lemma If  $A, B, C$  are sets such that  $B \cap C = \emptyset$ , then  $A \times B \cap A \times C = \emptyset$  as well and this extends to uncountably infinite case as well.

Now let  $x, y \in X$  be distinct. Then there exists (at least one) some  $j \in I$  our indexing set such that the components do not agree here, that is, for  $x_j, y_j \in X_j$  we have

$$x_j \neq y_j.$$

As  $X_j$  is Hausdorff, then there exists  $U_j, V_j \in \tau_j$  with  $x_j \in U_j, y_j \in V_j$  such that

$$U_j \cap V_j = \emptyset.$$

Then by the definition of the product topology, for each  $i \in I \setminus \{j\}$  we can take

$$U_i = V_i = X_i$$

And then define the neighborhoods of  $x, y$  as

$$U = \prod_{i \in I} U_i, V = \prod_{i \in I} V_i.$$

and since the  $U_i, V_i$  are disjoint at  $j$  together with our lemma we have that

$$U \cap V = \emptyset$$

with  $x \in U, y \in V$  thus  $(X, \tau_X)$  is Hausdorff as needed. □

(b) Prove  $Y = \prod_{n \in \mathbb{N}} [0, 1]_n$  is separable.

*Proof.* To show  $Y$  is separable, we construct a countable subset that is dense in  $Y$ . Consider the following subset of  $Y$ ,

$$A := \{(a_1, a_2, \dots) \in Y \mid \exists N \in \mathbb{N} \ni a_i \in \mathbb{Q} \cap [0, 1], 1 \leq i < N\}$$

I.e., all sequences with finitely many rational coordinates. I claim  $A$  is dense in  $Y$ , to see this we show every open set of  $Y$  intersects  $A$  nontrivially. Let  $x \in U \in \tau_Y$ . If  $x \in A$  then we are done as

$x \in U \cap A$ . Suppose  $x \in Y \setminus A$ . As we are in the product topology, our basis elements are of the form

$$\prod_{n=1}^{\infty} U_n \subset U$$

Where  $U_n = [0, 1]$  for all but finitely many  $n$ . And for those finitely many  $n$  we have the proper containment  $U_n \subset [0, 1]$ . If we let  $x = (x_1, x_2, \dots)$ , then for finitely many values, call them  $i$ , we have

$$x_i \in U_i \subset [0, 1]$$

As  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , we have for these finite many  $i$ , there exists some rational  $q_i \in \mathbb{Q}$  such that

$$q_i \in U_i$$

Then consider the point  $y = (y_1, y_2, \dots)$  such that

$$y_i = q_i; i \leq n, y_n = x_n; \forall n > i$$

Then  $y \in A$  and since  $U$  was arbitrary we have that  $A$  is dense in  $Y$ . □

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## 8. Continuity When restricted

(a) Let  $(X, \tau_X)$  be a topological space. If we can write  $X = \bigcup_{n \in \mathbb{N}} W_n$  where for each  $n \in \mathbb{N}$

$$f|_{W_n} : W_n \rightarrow Y$$

is continuous, then

$$f : X \rightarrow Y$$

is continuous.

*Proof.* Let

$$f : X \rightarrow Y$$

be a map of topological spaces with topologies  $\tau_X, \tau_Y$  respectively. Let  $V \in \tau_Y$ . Note that

$$\begin{aligned} f|_{W_k}^{-1}(V) &= f^{-1}(V) \cap W_k \\ &\in \tau_X. \end{aligned}$$

As  $f|_{W_n}$  is continuous for every  $n$ . But then we can write

$$\begin{aligned} f^{-1}(V) &= \bigcup_{k \in \mathbb{N}} f^{-1}(V) \cap W_k \\ &\in \tau_X \end{aligned}$$

as any union of open need be open thus  $f$  is continuous.  $\square$

(b) Let  $(X, \tau_X)$  be a topological space. If  $X = A \cup B$  where  $A, B$  are closed and  $f|_A : A \rightarrow Y, f|_B : B \rightarrow Y$  are continuous. Prove

$$f : X \rightarrow Y$$

is continuous.

*Proof.* Let  $V \subset Y$  be closed, then just as before, since  $A$  is a subset of  $X$ , then for all subsets of  $Y$  which  $V$  is, we have

$$f^{-1}(V) \cap A = (f|_A)^{-1}(V)$$

Holds for when  $x \in A$  and  $f(x) \in V$ . As  $V \subset Y$  is closed and the restrictions are continuous, we have that

$$\begin{aligned} &(f|_A)^{-1}(V) \cap A \\ &(f|_B)^{-1}(V) \cap B \end{aligned}$$

Are both closed in  $A, B$  respectively thus both are closed in  $X$  as well and we have

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \cap (A \cup B) \\ &= (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B) \end{aligned}$$

Which is a finite union of closed thus  $f^{-1}(V) \subset X$  is closed as needed thus  $f$  is continuous.  $\square$

(c) Assume  $X = \bigcup_{k=1}^{\infty} E_k$  where the  $E_k$  are all closed in  $X$  such that each  $E_k \rightarrow Y$  is continuous, is  $X \rightarrow Y$  also continuous?

*Proof.* False. Consider the map

$$f : \mathbb{Z} \rightarrow \mathbb{R}$$

where  $\mathbb{Z}$  is endowed with cofinite topology and  $\mathbb{R}$  has the standard topology. Let

$$\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} \{n\}$$

So our  $E_k$  are just singletons of integers, then  $f|_{E_k}$  is continuous for each  $k$ .

To see this, take  $C \subset \mathbb{R}$  closed, then we have

$$f|_{E_k}^{-1}(C) = f^{-1}(C) \cap E_k$$

which is either just a singleton or the empty set both of which are closed in  $\mathbb{Z}$  with cofinite topology.

On the other hand,  $f$  is not continuous as any open set  $(a, b) \subset \mathbb{R}$  has a pull back with infinite complement thus not open in  $\mathbb{Z}$  with cofinite topology.  $\square$

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## 9. Product Topology

(a) Define the product topology on the product  $X = \prod_{j=1}^{\infty} X_j$ .

*Proof.* The product topology on  $X$  is the product

$$\prod_{j=1}^{\infty} U_j$$

Where  $U_j \subseteq X_j$  are open and

$$U_j = X_j$$

$\forall$  but finitely many  $j$ , for finite  $j$ ,

$$U_j \subsetneq X_j$$

proper subset. More formally, if  $\pi_{\beta}$  is projection onto the  $\beta$ th coordinate, then

$$\mathcal{S}_{\beta} = \{\pi_{\beta}^{-1}(U_{\beta}) \mid U_{\beta} \text{ is open in } X_{\beta}\}$$

Then the topology generated by  $\bigcup_{\beta \in J} \mathcal{S}_{\beta}$  is the product topology.  $\square$

(b) Show the projection map  $p_i : X_1 \times X_2 \rightarrow X_i$  is an open map for  $i = 1, 2$ .

*Proof.* For each  $i = 1, 2$ , let  $\tau_i$  denote the topology on  $X_i$ . Let  $U \in \tau_{\alpha}$ , then by the definition of the product topology we can write

$$U = \bigcup_{j \in J} \bigcap_{k=1}^{n_j} p_{i_{k,j}}^{-1} U_{k,j},$$

where  $J$  is an arbitrary indexing set,  $n_j \in \mathbb{N}$  and  $i_{k,j} = 1, 2$ . Then for every  $i = 1, 2$ , define  $V_{i,k,j} \in \tau_i$  via

$$V_{i,k,j} = \begin{cases} U_{k,j} & ; i = i_{k,j} \\ X_i & ; i \neq i_{k,j} \end{cases}$$

By the definition of projection we have

$$p_{i_{k,j}}^{-1}(U_{k,j}) = V_{1,k,j} \times V_{2,k,j}.$$

And without any loss of generality we can suppose  $i = 1$  and compute

$$\begin{aligned} p_1(U) &= \bigcup_{j \in J} p_1\left(\bigcap_{k=1}^{n_j} p_{i_{k,j}}^{-1}(U_{k,j})\right) \\ &= \bigcup_{j \in J} p_1\left(\bigcap_{k=1}^{n_j} (V_{1,k,j} \times V_{2,k,j})\right) \\ &= \bigcup_{j \in J} p_1\left(\bigcap_{k=1}^{n_j} V_{1,k,j} \times \bigcap_{k=1}^{n_j} V_{2,k,j}\right) \\ &= \bigcup_{j \in J} \bigcap_{k=1}^{n_j} V_{1,k,j} \\ &\in \tau_1 \end{aligned}$$

and thus  $p_1$  is an open map. The same proof works for  $p_2$ .  $\square$

(c) If  $Y$  is Hausdorff and

$$f : X \rightarrow Y$$

is continuous, prove the graph

$$\Delta = \{(x, f(x)) | x \in X\}$$

is closed in  $X \times Y$ .

*Proof.* We show  $\Delta^c$  is open instead.

Let  $(x, y) \in X \times Y \setminus \Delta$ , then  $y \neq f(x)$ .

As  $y, f(x) \in Y$  which is Hausdorff they can be separated via open sets of  $Y$ .

That is,  $\exists U, V \subset Y$  open with  $y \in U, f(x) \in V$  s.t.

$$U \cap V = \emptyset$$

By Munkres Theorem 18.1(4) since  $f$  is continuous,  $\exists W \subseteq X$  with  $x \in W$  s.t.

$$f(W) \subseteq V$$

Then  $W \times U$  is an open neighborhood of  $(x, y)$  disjoint from  $\Delta$  thus  $X \times Y \setminus \Delta = \Delta^c$  is open, and therefore  $\Delta$  is closed.  $\square$

(d) If  $Y$  is Hausdorff and

$$f : X \rightarrow Y$$

is continuous, prove

$$G : X \rightarrow X \times Y$$

defined via

$$G(x) = (x, f(x))$$

is a closed map.

*Proof.* Let  $C \subseteq X$  be closed, we wish to show

$$G(C) \subset X \times Y$$

is closed. Let  $(x, y) \in X \times Y \setminus G(C)$ , then  $y \neq f(x)$  which are both in  $Y$ .

Since  $Y$  is Hausdorff,  $\exists U, V \subseteq Y$  both open with  $y \in U, f(x) \in V$  such that

$$U \cap V = \emptyset$$

As  $f$  is continuous however,  $\exists W \subseteq X$  an open neighborhood of  $x$  such that

$$f(W) \subseteq V$$

Then  $W \times U$  is an open neighborhood of  $(x, y)$  disjoint from  $G(C)$ , thus  $G(C)^c$  is open forcing  $G(C)$  to be closed  $\therefore G$  is a closed map.  $\square$

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## 10. Homeomorphisms

(a) Prove  $(0,1)$  with the subspace topology is homeomorphic to  $\mathbb{R}$  with the standard topology.

*Proof.* Let

$$f : (0,1) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

be defined via

$$f(x) := \pi x - \frac{\pi}{2}$$

and let

$$g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$$

be defined via

$$g(x) := \tan x$$

Then

$$h : (0,1) \rightarrow \mathbb{R}$$

defined via

$$\begin{aligned} h(x) &:= g(f(x)) \\ &= \tan\left(\pi x - \frac{\pi}{2}\right) \end{aligned}$$

is the desired homeomorphism with inverse defined via

$$h^{-1}(x) := \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2}$$

By calculus we are done. □

(b) Assume  $X, Y$  are metric spaces that are homeomorphic. Prove or give counterexample:  $X$  complete implies  $Y$  complete, that is, completeness preserved under cont?

*Proof.* Part (a) □

(c) Prove  $[a, b] \not\cong (c, d)$ .

*Proof.* First a small lemma (proof left to the interested reader; HINT: First restrict the domain, then restrict the range.)

Lemma: If

$$f : X \rightarrow Y$$

is a continuous map of topological spaces, then for any  $x \in X$ ,

$$\bar{f} : X \setminus \{x\} \rightarrow Y \setminus \{f(x)\}$$

is continuous as well. Moreover, if  $f$  is a homeomorphism, then  $\bar{f}$  is a homeomorphism as well.

Now let us assume  $[a, b] \cong (c, d)$ . Then there exists a homeomorphism

$$g : [a, b] \rightarrow (c, d).$$

By our lemma above

$$\bar{g} : [a, b] \setminus \{a\} \rightarrow (c, d) \setminus \{g(a)\}$$

is a homeomorphism as well. I.e.,

$$g : (a, b) \rightarrow (c, g(a)) \cup (g(a), d)$$

is a homeomorphism. Note that the domain is still connected while the range space is clearly disconnected and since connectedness is a topological property, this contradicts continuity of  $\bar{g}$  and thus  $[a, b] \not\cong (c, d)$  as needed. One can check that the range space in fact has a separation.  $\square$

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## 11. Topology of finite point-set

Let  $X = \{1, 2, 3, 4\}$  be given by the topology  $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

(a) Show any two disjoint closed sets have disjoint open neighborhoods

*Proof.* By definition, the closed sets in  $X$  are taken to be the complements of the given open sets, that is,

$$\mathcal{C} = \{X, \emptyset, \{2, 3, 4\}, \{3, 4\}, \{2, 4\}, \{4\}\}$$

are all of the closed subspaces of  $X$ . As 4 is an element of every member of  $\mathcal{C}$  except  $\emptyset$  and  $\emptyset$  is disjoint with every non-empty set, the pairs of disjoint closed subspaces are

$$X, \emptyset$$

$$\{2, 3, 4\}, \emptyset$$

$$\{3, 4\}, \emptyset$$

$$\{2, 4\}, \emptyset$$

$$\{4\}, \emptyset$$

where the neighborhood  $X$  contains each closed set except for  $\emptyset$  and clearly  $\emptyset$  is its own neighborhood which is disjoint from  $X$  as needed.  $\square$

(b) Show  $(X, \tau)$  is not  $T_1$

*Proof.* Consider the elements 1, 2. The open neighborhoods of 2 are

$$\{1, 2\}, \{1, 2, 3\}.$$

Since both of these contain 1 we cannot find open sets for 1 and 2 that do not contain each other thus  $(X, \tau)$  is not  $T_1$ .  $\square$

(c) Let  $A = \{1, 2, 3\} \subset X$  be endowed with the subspace topology. Find disjoint closed subsets of  $A$  that do not have disjoint neighborhoods.

*Proof.* As  $A$  is endowed with the subspace topology we can write out the topology on  $A$  as follows

$$\tau_A = \{\emptyset, A, \{1\}, \{1, 2\}, \{1, 3\}\}.$$

Then the closed subsets of  $A$  are given by

$$\mathcal{C}_A = \{A, \emptyset, \{2, 3\}, \{3\}, \{2\}\}.$$

Then  $\{2\}, \{3\}$  are disjoint closed sets in  $A$ . The neighborhoods of  $\{2\}$  are

$$A, \{1, 2\},$$

and the neighborhoods of  $\{3\}$  are

$$A, \{1, 3\}.$$

And since  $\{1, 2\} \cap \{1, 3\} = \{1\}$  is non-empty, we have found disjoint closed subsets of  $A$  with non disjoint neighborhoods.  $\square$

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## 12. Closures connected/path

(a) If  $X \subseteq Z$  is a connected subset of a topological space, show that  $\overline{X} \subseteq Z$  is connected as well.

*Proof.* Let  $X \subseteq Z$  be a connected subspace of a topological space. Suppose towards a contradiction that  $\overline{X}$  is not connected. Then there exists a separation of  $\overline{X}$ , that is,

$$\overline{X} = A \cup B,$$

$A, B \in \tau_{\overline{X}}$  non-empty and disjoint. As  $X$  is connected, by the connected Lemma we have WLOG that  $X = A \cap X$ . As  $B$  is non-empty, it contains some  $b$ , namely  $b$  is a limit point of  $X$  and thus  $B$  is an open set containing a limit point of  $X$  thus it must intersect  $X \subset A$  non-trivially contradicting

$$A \cap B = \emptyset.$$

Thus  $\overline{X}$  is connected. □

(b) Show (a) fails for path-connected subspaces.

*Proof.* Take the topologist's sin curve. That is, the function

$$f : \mathbb{R}^+ \rightarrow [-1, 1]$$

which is defined via

$$x \mapsto \sin\left(\frac{1}{x}\right).$$

Then  $\{(x, y) \mid y = \sin(\frac{1}{x})\}$  is path-connected, however the closure given via

$$\{(x, y) \in \mathbb{R}^2 \mid y = \sin\left(\frac{1}{x}\right)\} \cup \{0\} \times [-1, 1]$$

is not path-connected. □

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### 13. Cofinite Topology

Let  $\tau$  be the finite complement topology on  $\mathbb{R}$ . That is,  $U \subseteq \mathbb{R}$  is open if and only if  $U$  is empty or  $\mathbb{R} \setminus U$  is finite.

(a) Is  $(\mathbb{R}, \tau)$  Hausdorff?

*Proof.* I claim  $(\mathbb{R}, \tau)$  is not Hausdorff. Let  $x, y \in \mathbb{R}$  and  $U \in \tau$  be a neighborhood of  $x$ . Then  $\mathbb{R} \setminus U$  is finite so we can write

$$\mathbb{R} \setminus U = \{p_1, \dots, p_n\}.$$

Similarly we can let  $V \in \tau$  be a neighborhood of  $y$  and write

$$\mathbb{R} \setminus V = \{q_1, \dots, q_m\}.$$

We would like for  $U, V$  to have a non-empty intersection. As  $\mathbb{R}$  is infinite, we can find a  $z \in \mathbb{R}$  such that  $z \neq p_i$  for  $1 \leq i \leq n$  and  $z \neq q_j$  for  $1 \leq j \leq m$ . This would imply  $z \in U \cap V$  and so  $(\mathbb{R}, \tau)$  is not Hausdorff.  $\square$

(b) Is  $(\mathbb{R}, \tau)$  compact?

*Proof.* I claim  $(\mathbb{R}, \tau)$  is compact. Let  $\{U_\alpha\}_{\alpha \in A}$  be an arbitrary open cover for  $\mathbb{R}$ , that is

$$\mathbb{R} \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

For each  $\alpha \in A$  we know  $\mathbb{R} \setminus U_\alpha$  is finite, in particular we can take some  $\beta \in A$  and so  $\mathbb{R} \setminus U_\beta$  is finite. Then for each  $x \in \mathbb{R} \setminus U_\beta$  let  $U_x \in \tau$  be the neighborhood containing  $x$ . Then it follows that

$$\mathbb{R} \subseteq U_\beta \cup \{U_x \mid x \in \mathbb{R} \setminus U_\beta\}$$

is a finite sub-cover of our arbitrary covering thus  $(\mathbb{R}, \tau)$  is Hausdorff as needed.  $\square$

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## 14. Sequence of closed

Let  $F_1, F_2, \dots$  be a sequence of closed subsets of a topological space  $X$ . Suppose that for each  $x \in X$  we can find a neighborhood of  $x$  say  $N_x$  such that  $N_x \cap F_j \neq \emptyset$  for only finitely many  $j$  values. Prove  $\bigcup_{j=1}^{\infty} F_j$  is closed.

*Proof.* We will show the complement with respect to the entire space is open. That is, we show

$$X \setminus \bigcup_{j=1}^{\infty} F_j$$

Let  $x_0 \in X \setminus \bigcup_{j=1}^{\infty} F_j$  be arbitrary. We must find a neighborhood of  $x$  that is disjoint from  $\bigcup_{j=1}^{\infty} F_j$ . As  $x_0 \in X$  we are guaranteed the existence of neighborhood  $N_{x_0}$  such that

$$N_{x_0} \cap F_j \neq \emptyset$$

for finitely many  $j$  values. That is, there exists a finite set  $J$  such that

$$N_{x_0} \bigcap_{j \in J} F_j \neq \emptyset.$$

Then I claim the open neighborhood of  $x_0$  that is disjoint from  $\bigcup_{j=1}^{\infty} F_j$  is given by the intersection

$$N_{x_0} \bigcap_{j \in J} X \setminus F_j.$$

Clearly we have that  $x_0 \in N_{x_0} \bigcap_{j \in J} X \setminus F_j$  and is clearly disjoint from  $\bigcup_{j=1}^{\infty} F_j$ . Moreover, as  $\tau_X$  is closed under finite intersecions we have that  $N_{x_0} \bigcap_{j \in J} X \setminus F_j \in \tau_X$  as needed for our neighborhood of  $x_0$  and so  $X \setminus \bigcup_{j=1}^{\infty} F_j$  is open thus  $\bigcup_{j=1}^{\infty} F_j$  is closed.  $\square$

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## 15 (Incomplete)

Let  $(X, \tau)$  be a topological space and let  $\mathcal{C}$  be the collection of closed sets. A *filter* on  $\mathcal{C}$  is a collection  $\mathcal{F}$  of sets from  $\mathcal{C}$  such that (1)  $\emptyset \notin \mathcal{F}$ , (2) If  $C_1, C_2 \in \mathcal{F}$ , then  $C_1 \cap C_2 \in \mathcal{F}$ , (3) If  $C_1 \subset C_2$  with  $C_1 \in \mathcal{F}$  and  $C_2 \in \mathcal{C}$ , then  $C_2 \in \mathcal{F}$ . Show that if each filter on  $\mathcal{C}$  has non-empty intersection, then  $(X, \tau)$  is compact.

*Proof.* Let  $\{U_\alpha\}_{\alpha \in A}$  be an arbitrary open cover for  $X$ . That is,

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha,$$

where  $U_\alpha \in \tau$  for each  $\alpha$ . Then by definition we have for each  $\alpha \in A$  that  $X \setminus U_\alpha \in \mathcal{C}$ . Let  $\mathcal{B} = \{B \in \tau \mid B \subseteq \bigcup_{i \in I} U_{\alpha_i}\}$  for some finite  $I \subset A$ . Then  $X \setminus B \in \mathcal{C}$  for each  $B \in \mathcal{B}$ . I claim that

$$\mathcal{F}_\beta = \{X \setminus B \mid B \in \mathcal{B}\}$$

defines a filter on  $\mathcal{C}$ . For (1), let us suppose  $\emptyset \in \mathcal{F}_\beta$ , then  $\emptyset = X \setminus B$  for some  $B \in \mathcal{B}$  which implies  $B = X$  and since  $B \in \mathcal{B}$ , we have that  $X \subseteq \bigcup_{i \in I} U_{\alpha_i}$ .  $\square$

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## 16. Connected Lemma

Let  $(X, \tau)$  be a topological space and  $Y \subset X$  a connected subspace endowed with the subspace topology. If  $A \cup B$  forms a separation of  $X$ , then  $Y \subset A$  or  $Y \subset B$ .

*Proof.* As  $A \cup B$  forms a separation, we have that

$$X = A \cup B$$

where  $A, B \in \tau$  are both non-empty and disjoint as a pair. Let us suppose towards a contradiction that there exists  $a, b \in Y$  such that  $a \in A$  and  $b \in B$ . I claim then that

$$\begin{aligned} Y \cap X &= Y \cap (A \cup B) \\ &= (Y \cap A) \cup (Y \cap B) \end{aligned}$$

forms a separation of  $Y$ . Since  $Y$  is endowed with the subspace topology and  $A, B \in \tau$  we have that  $Y \cap A, Y \cap B \in \tau_Y$ . That is, they are both open in  $Y$ . If they were not disjoint then there would exist some  $\alpha$  such that

$$\alpha \in (Y \cap A) \cap (Y \cap B)$$

contradicting  $A, B$  being disjoint as a pair and thus  $Y \cap A, Y \cap B$  are disjoint as well. Lastly we know by existence of  $a, b$  that they are non-empty thus together they form a separation of  $Y$  which is connected a contradiction thus  $Y$  must lie entirely within  $A$  or  $B$ .  $\square$

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## 17. Arbitrary collection of topologies on a set

(a) Let  $X$  be set and  $\{\tau_\alpha\}_{\alpha \in A}$  be a collection of topologies on  $X$ . Prove  $\bigcap_{\alpha \in A} \tau_\alpha$  is a topology on  $X$ .

*Proof.* Suppose for each  $\alpha \in A$  we have that  $(X, \tau_\alpha)$  is a topological space, we must show  $(X, \bigcap_{\alpha \in A} \tau_\alpha)$  is also a topological space. First we check  $\emptyset, X \in \bigcap_{\alpha \in A} \tau_\alpha$ . Since for each  $\alpha \in A$ ,  $\tau_\alpha$  is a topology on  $X$ , we have for every  $\alpha \in A$  that  $\emptyset, X \in \tau_\alpha$  and thus

$$\emptyset, X \in \bigcap_{\alpha \in A} \tau_\alpha$$

as needed. Next suppose that

$$U_1, U_2, \dots, U_n \in \bigcap_{\alpha \in A} \tau_\alpha.$$

Then for every  $\alpha \in A$  we have

$$U_1, U_2, \dots, U_n \in \tau_\alpha.$$

As for each  $\alpha \in A$ ,  $\tau_\alpha$  is a topology on  $X$ , by the closure property we get

$$\bigcap_{i=1}^n U_i \in \bigcap_{\alpha \in A} \tau_\alpha.$$

Lastly we must check arbitrary unions are closed. That is if for every  $\beta \in B$  some indexing set let us suppose

$$U_\beta \in \bigcap_{\alpha \in A} \tau_\alpha.$$

Then for every  $\alpha \in A$

$$U_\beta \in \tau_\alpha$$

which are each a topology as noted before thus by closure property of arbitrary unions we get (for every  $\alpha \in A$ , that is.)

$$\bigcup_{\beta \in B} U_\beta \in \tau_\alpha,$$

which gives us

$$\bigcup_{\beta \in B} U_\beta \in \bigcap_{\alpha \in A} \tau_\alpha,$$

as needed making  $(X, \bigcap_{\alpha \in A} \tau_\alpha)$  a topological space.  $\square$

(b) Given an example to show  $\bigcup_{\alpha \in A} \tau_\alpha$  is not necessarily a topology given  $\tau_\alpha$  is a topology for each  $\alpha \in A$ .

*Proof.* Let our set  $X$  be given as the following three point set

$$X = \{a, b, c\}.$$

Consider the following two topologies on  $X$ ,

$$\tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{b\}\}.$$

Then their union is given by

$$\tau_1 \cup \tau_2 = \{\emptyset, X, \{a\}, \{b\}\}$$

Which is not closed under even finite unions as

$$\{a\} \cup \{b\} \notin \tau_1 \cup \tau_2$$

Thus unions of topologies need not be a topology. □

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## 18. Intervals in $\mathbb{R}$ are connected

Prove intervals in  $\mathbb{R}$  are connected.

*Proof.* Let  $I \subset \mathbb{R}$  be an interval. To show  $I$  is connected, we assume towards a contradiction that  $I$  is disconnected. That is, there is a separation of the interval

$$I = A \cup B$$

Where  $A, B \in \tau_I$  (the topology on  $I$  when given the subspace topology inherited from  $\mathbb{R}_{\text{standard}}$ ), non-empty and disjoint. Thus we are guaranteed existence of  $a \in A$  and  $b \in B$  such that

$$a \notin B, b \notin A.$$

let  $I_0 = [a, b]$ . Note that  $I_0 \subseteq I$ . Then we can define

$$A_0 = A \cap I_0, B_0 = B \cap I_0.$$

Then  $A_0 \cup B_0$  forms a separation of  $I_0$ . To see this we already know they are non-empty by the existence of  $a, b$ . If they were not disjoint then there exists some  $\alpha \in A \cap I_0 \cap B$  contradicting  $A, B$  being disjoint as they form a separation of  $I$ . Lastly since  $A, B \in \tau_I$  we have that (in the subspace topology)  $A \cap I_0, B \cap I_0 \in \tau_{I_0}$ . Are both open. Note that  $A_0 \subset \mathbb{R}$  is non-empty thus it inherits the least upper bound property so we can define

$$c := \sup(A_0).$$

However  $A_0$  is closed because  $B_0$  is open thus  $c \in \overline{A_0}$ . and so  $c \notin B_0$ . As  $c$  is the supremum of  $A_0$ , for any  $x \in I_0$  with  $c < x$  we have that  $x \notin A_0$  thus we get

$$(x, b] \subset B_0.$$

But then  $c$  (Keep in mind that  $c \in I_0$ ) becomes a limit point of  $B_0$  forcing  $c \in \overline{B_0}$  and since  $A_0 \cup B_0$  form a separation of  $I_0$ ,  $c \notin A_0$  contradicting

$$c \in I_0 = A_0 \cup B_0.$$

thus  $I$  is connected as there is no separation. □

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## 19. Subspace of separable is separable

If  $X$  is a separable metric space, then so is any subspace  $Y$ .

*Proof.* Let  $(X, \tau)$  be a topological space. Assume that  $X$  is separable, then we show for any subspace  $Y \subseteq X$  that  $Y$  is separable. If  $Y = X$  or  $\emptyset$  we are done so we suppose  $\emptyset \neq Y \subsetneq X$ . We must construct a countable dense subset for  $Y$ . Let  $A \subset X$  be a countable dense subset. Then we can write

$$A = \{a_1, a_2, \dots\}$$

such that

$$\overline{A} = X.$$

That is, for every  $x \in X$  and  $U_x \in \tau$  containing  $x$  we have

$$U_x \setminus \{x\} \cap A \neq \emptyset.$$

If we take  $y \in Y$ , then for any given  $\varepsilon > 0$  we have that

$$B_\varepsilon(y) \cap A \neq \emptyset.$$

As the intersection is non-empty let  $x_k \in B_\varepsilon(y) \cap A$ . I.e.,

$$x_k \in \{B_\varepsilon(y) \cap A : \varepsilon \in \mathbb{R}^+\}.$$

Thus

$$B_\varepsilon(x_k) \cap Y \neq \emptyset.$$

Then we can take

$$\mathcal{B} = \{(k, \varepsilon) : B_\varepsilon(x_k) \cap Y \neq \emptyset\},$$

which is non-empty. So for each  $(k, \varepsilon)$  take  $y_{k,\varepsilon} \in B_\varepsilon(x_k) \cap Y \neq \emptyset$  and let

$$Z = \{y_{k,\varepsilon} : (k, \varepsilon) \in \mathcal{B}\}.$$

And so we have that  $Z \subset Y$  is countable since the elements are pulled from elements who are in  $A$  which is countable. We must show  $Z$  is dense in  $Y$ . That is, we must show

$$\overline{Z} = Y.$$

Let  $y \in Y$  and  $r > 0$  and choose  $\varepsilon$  such that

$$\varepsilon \leq \frac{r}{2}.$$

Then we can always find a  $k \in \mathbb{N}$  such that

$$x_k \in B_\varepsilon(y).$$

Then  $(k, \varepsilon) \in \mathcal{B}$  and by the triangle inequality,

$$\begin{aligned} d(y, y_{k,\varepsilon}) &\leq d(y, x_k) + d(x_k, y_{k,\varepsilon}) \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &\leq r. \end{aligned}$$

Thus  $y_{k,\varepsilon} \in B_r(y)$  and thus  $y \in \overline{Z}$  making  $Z$  dense in  $Y$  so  $Y$  is separable. □

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## 20. Hausdorff Diagonal

Let  $(X, \tau)$  be a topological space. Then  $X$  is Hausdorff if and only if  $\Delta = \{(x, x) : x \in X\}$  is closed in  $X \times X$ .

*Proof.* First let us assume that  $\Delta$  is closed in  $X \times X$ . We wish to show  $X$  is Hausdorff so let  $x, y \in X$  be arbitrary. As  $\Delta \subset X \times X$  is closed, by definition we know that  $\Delta^c \in \tau$ . Then there exists a basis element of the form

$$U \times V \quad ; U, V \in \tau,$$

such that

$$(x, y) \in U \times V \subset \Delta^c.$$

And so we have that

$$(U \times V) \cap \Delta = \emptyset$$

which gives us  $x \in U, y \in V$ . Lastly, I claim that

$$U \cap V = \emptyset.$$

If not, then there exists some  $z \in U \cap V$  forcing

$$(z, z) \in U \times V.$$

Moreover,  $(z, z) \in \Delta$  by definition, contradicting disjointness of  $U \times V$  and  $\Delta$  and so  $X$  is Hausdorff. On the other hand, let us assume that  $X$  is Hausdorff and we wish to show that  $\Delta \subset X \times X$  is closed. We show this by showing  $\Delta^c$  is open. Let  $x, y \in \Delta^c$ . As  $X$  is Hausdorff we are guaranteed the existence of  $U_x, U_y \in \tau$  such that

$$U_x \cap U_y = \emptyset.$$

I claim that

$$(U_x \times U_y) \cap \Delta.$$

Let  $(a, b) \in (U_x \times U_y) \cap \Delta$  then  $a = b \in U_x \cap U_y$  a contradiction and thus  $\Delta^c \in \tau$  and so  $\Delta \subset X \times X$  is closed.  $\square$

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## 21. Equivalence of continuity in metric space

A function  $f$  is continuous using open sets if and only if it is continuous in the  $\varepsilon - \delta$  sense. Let  $\tau_X, \tau_Y$  denote topologies in  $X$  and  $Y$  respectively.

*Proof.* Let  $X, Y$  be metric spaces and

$$f : X \rightarrow Y$$

a map between them. First we will assume  $f$  is open set continuous. That is, if  $V \in \tau_Y$  then  $f^{-1}(V) \in \tau_X$ . Let  $x_0 \in X$  and  $\varepsilon > 0$  be given. Then  $f(x_0) \in Y$  and we have that

$$(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \in \tau_Y.$$

Then since  $f$  is continuous we get

$$x_0 \in f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)) \in \tau_X.$$

So we can find a basis element containing  $x_0$  fully contained in  $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ . I.e., there exists  $\delta > 0$  such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)).$$

But then we have

$$f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

Thus for any  $\varepsilon > 0$  we can always find a  $\delta > 0$  such that if

$$|x - x_0| < \delta$$

then

$$|f(x) - f(x_0)| < \varepsilon$$

making  $f$  continuous in the  $\varepsilon - \delta$  senses.

On the other hand suppose that  $f$  is continuous in the  $\varepsilon - \delta$  sense and let

$$f : X \rightarrow Y$$

be our map. Let  $V \in \tau_Y$ . We wish to show  $f^{-1}(V) \in \tau_X$ . Let  $x_0 \in f^{-1}(V)$ , then  $f(x_0) \in V \in \tau_Y$  and so there exists an  $\varepsilon > 0$  such that

$$f(x_0) - \varepsilon, f(x_0) + \varepsilon \subset V.$$

And since  $f$  is continuous in  $\varepsilon - \delta$  sense are guaranteed the existence of some  $\delta > 0$  such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}(V).$$

This  $\delta$  ball is the neighborhood of  $x_0$  properly contained in  $f^{-1}(V)$  thus  $f^{-1}(V) \in \tau_X$  as needed.  $\square$

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## 22. Continuous function over compact space

(a) Show continuous over compact attain a max.

*Proof.* Let  $f : X \rightarrow \mathbb{R}$  be continuous. Suppose that  $X$  is compact. Since compactness is a topological property,  $f(X)$  is compact in  $\mathbb{R}$ . By Heine-Borel, subsets of  $\mathbb{R}$  under the standard topology are compact if and only if they are both closed and bounded. Thus  $f(X) \subset \mathbb{R}$  is closed and bounded. Since  $f(X)$  is bounded there exists some  $M \in \mathbb{R}$  such that for every  $x \in X$  we have that

$$|f(x)| \leq M.$$

This tells us the supremum not only exists but is finite thus we can define

$$a := \sup_{x \in X} f(x)$$

Making  $a$  a limit point of  $f(X)$  which is closed in  $Y$  forcing  $a \in f(X)$  then by definition we have that

$$f(x) \leq a$$

for every  $f(x) \in f(X)$ . Hence  $f(X)$  attains its max. □

(b) Show continuous over compact is uniform.

*Proof.* Let

$$f : X \rightarrow Y$$

be a map between metric spaces. If  $f$  is continuous and  $X$  is compact, prove that  $f$  is uniformly continuous. I.e.,  $\delta$  is not dependant on each point. Since  $f$  is continuous, for each  $x \in X$  and any given  $\varepsilon > 0$ , there is a  $\delta_x$  such that if

$$d_X(x, y) < \delta_x,$$

then

$$d_Y(f(x), f(y)) < \varepsilon.$$

In other words

$$f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x)). \quad (*)$$

We now have that  $\{B_{\frac{\delta_x}{2}}(x)\}_{x \in X}$  is an open cover for  $X$ . As  $X$  is compact, we can find a finite subset  $A \subset X$  such that

$$X \subseteq \bigcup_{x \in A} B_{\frac{\delta_x}{2}}(x).$$

Then we can take our  $\delta$  to be

$$\delta = \min_{x \in A} \left( \frac{\delta_x}{2} \right)$$

Then we have that  $d_X(x, y) < \delta$ , then since  $x \in B_{\frac{\delta_x}{2}}(x)$  we have that  $y \in B_{\delta_x}(x)$  (keep in mind  $x \in A$ ). Lastly, if  $d_X(x, y) < \delta$  I claim  $d_Y(f(x), f(y)) < \varepsilon$ . Applying (\*) we get that

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(z)) + d_Y(f(z), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

as needed. □

Alternate proof:

*Proof.* Let

$$f : X \rightarrow Y$$

be a map between metric spaces. If  $f$  is continuous and  $X$  is compact, then we show  $f$  is uniformly continuous. We say  $f$  is continuous in the  $\varepsilon - \delta$  at  $x \in X$  if for any  $\varepsilon > 0$  we can find a  $\delta > 0$  such that if

$$d_X(x, y) < \delta,$$

then

$$d_Y(f(x), f(y)) < \varepsilon.$$

I claim that if  $f$  is  $\frac{\varepsilon}{2} - \delta$  continuous, then  $f$  is  $\varepsilon - \frac{\delta}{2}$  continuous. To see this, note that for every  $x' \in B_{\frac{\delta}{2}}(x)$  and  $y \in B_{\frac{\delta}{2}}(x')$  we have that  $x', y \in B_{\delta}(x)$ . Thus we can compute

$$\begin{aligned} d_Y(f(x'), f(y)) &\leq d_Y(f(x'), f(x)) + d_Y(f(x), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

And we have proven the claim. Let  $\varepsilon > 0$  and  $x \in X$ . By  $\varepsilon - \delta$  continuity, there is some  $n \in \mathbb{N}$  such that  $f$  is  $\frac{\varepsilon}{2} - \frac{1}{n}$  continuous. Then by the claim,  $f$  is  $\varepsilon - \frac{1}{2n}$  continuous. Moreover, as  $X$  is compact, it can be covered by a finite number of these balls so let  $n_0$  be the max  $n$  value in the finite collection, then  $f$  is  $\varepsilon - \frac{1}{2n_0}$  continuous on every neighborhood of  $X$  and thus on all of  $X$ .  $\square$

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## 23. Countable product of separable

Prove the countable product of separable is separable.

*Proof.* Let  $\{X_n\}_{n \in \mathbb{N}}$  be a collection of separable metric spaces. Let  $D_n$  be the associated countable dense subset and fix  $x_n \in D_n$ . Then for each  $m \in \mathbb{N}$ , we can define

$$\begin{aligned} E_m &= \{y \in D_n : y_n = x_n; \forall n \geq m\} \\ &= \prod_{1 \leq n < m} D_n \times \prod_{n \geq m} \{x_n\}. \end{aligned}$$

Which is clearly countable and thus  $\bigcup_m E_m$  is countable as well. I claim that  $\bigcup_m E_m$  is dense in  $\prod_n X_n$ . Note by the definition of product topology we can find a basis element of the form

$$B = \prod_{1 \leq n < m} V_n \times \prod_{n \geq m} X_n.$$

Where  $V_n \subset X_n$  open. This is since for all but finitely many, the open sets are the whole space, in the product topology and thus

$$B \cap \bigcup_m E_m \neq \emptyset$$

forcing  $\bigcup_m E_m$  to be dense in  $\prod_n X_n$  as needed. □

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## 24. Closed intervals are compact

Prove for  $a < b \in \mathbb{R}$  that  $[a, b]$  is compact.

*Proof.* Let  $[a, b]$  for  $a < b$  real numbers be an interval. We would like to show it is compact. Let  $\{U_\alpha\}_{\alpha \in A}$  be an arbitrary open cover for  $[a, b]$  ( $A$  some arbitrary indexing set). That is,

$$[a, b] \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Let

$$F := \{x \in [a, b] \mid \exists B \subset A, \text{ finite } \ni [a, x] \subseteq \bigcup_{\alpha \in B} U_\alpha\}.$$

Note that  $F \neq \emptyset$ . This is because  $a \in F$  as the empty set is compact. We have that  $F$  is a non-empty subset of  $\mathbb{R}$  thus it inherits the least-upper-bound property. So we define

$$c := \sup F \in [a, b].$$

I claim that  $c = b$ . We know  $c > a$  because for  $\varepsilon > 0$  we there is a neighborhood  $U_i$  such that

$$[a, a + \varepsilon] \subset U_i$$

Thus  $x \geq a + \varepsilon$  and we know

$$a < c < b.$$

Now take  $\beta \in A$  such that  $c \in U_\beta$  and choose  $\varepsilon > 0$  such that

$$a \leq c - \varepsilon < c < c + \varepsilon \leq b$$

and

$$[c - \varepsilon, c + \varepsilon] \subset U_\beta$$

Since  $c - \varepsilon$  is not an upper bound of  $F$  there is some  $c_0$  with

$$c - \varepsilon \leq c_0 \leq c$$

such that  $c_0 \in F$ . which means  $[a, c_0]$  has a finite sub-cover from our original cover. I.e.,

$$[a, c_0] \subset \bigcup_{\alpha \in B} U_\alpha$$

which implies

$$[a, c + \varepsilon] \subset \bigcup_{\alpha \in B} U_\alpha \cup U_\beta$$

Forcing  $c + \varepsilon \in F$  contradicting the fact that  $c$  is the upper bound as  $c < c + \varepsilon$ . Thus  $c = \sup F = b$ . Lastly, we show  $b \in F$ . To see this, note for any  $\varepsilon > 0$  we know that there exists some  $\gamma \in A$  such that

$$[b - \varepsilon, b] \subset U_\gamma.$$

This gives us the existence of some  $c_0 \in [b - \varepsilon, b]$  such that  $c_0 \in F$ . Then we can write

$$\begin{aligned} [a, b] &= [a, c_0] \cup [b - \varepsilon, b] \\ &\subseteq \bigcup_{\alpha \in B} U_\alpha \cup U_\gamma. \end{aligned}$$

Thus  $b \in F$  and we have found a finite sub-cover for  $[a, b]$  as needed. □

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## 25. Successor Topology

Let  $X = \mathbb{N}$  and equip it with the following topology

$$\tau = \{U \subset \mathbb{N} : (2n - 1) \in U \Rightarrow 2n \in U\}.$$

Prove that  $(X, \tau)$  is locally compact but not compact.

*Proof.* For locally compact, let  $x \in \mathbb{N}$ . Then  $x$  is even or odd. If  $x$  is even, then take  $U$  to be

$$U := \{x, x + 1, x + 2\}$$

and take our compact superset of  $U$  to be

$$K := \{x, x + 1, x + 2, x + 3\}$$

Then  $U$  is open and we have

$$x \in U \subset K$$

and  $K$  is compact as it is a finite point set. Next suppose  $x$  is odd, then take  $U$  to be

$$U := \{x, x + 1\}$$

and our compact set  $K$  to be

$$K := \{x, x + 1, x + 2\}$$

Then we have that

$$x \in U \subset K$$

and  $(X, \tau)$  is locally compact.

For showing it is not compact, consider the open covering

$$\bigcup_{n \in \mathbb{N}} \{2n - 1, 2n\}$$

I claim this has no finite subcovering. If there was, say there existed some finite subset of  $\mathbb{N}$  call it  $A$  such that

$$\mathcal{A} = \bigcup_{n \in A} \{2n - 1, 2n\} \supset \mathbb{N}$$

Then there exists an  $N \in \mathbb{N}$  such that

$$N = \max_{n \in A} 2n$$

And

$$\{2N - 1, 2N\} \notin \mathcal{A}$$

but

$$2N - 1, 2N \in \mathbb{N}.$$

Thus  $(X, \tau)$  is not compact as needed. □

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## 26. Prime induced metric on $\mathbb{Z}$ .

Let  $p$  be an odd prime. And define  $d : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$  as  $d(m, n) = 0$  if  $m = n$  and

$$d(m, n) = \frac{1}{r+1},$$

where  $r$  is the largest positive integer such that

$$p^r \mid m - n$$

and we write

$$p^r \parallel m - n.$$

Show  $d$  induces a metric on  $\mathbb{Z}$ .

*Proof.* Clearly we have  $d(m, n) = 0$  by definition. Furthermoe, as  $r \in \mathbb{Z}^+$ , we have that

$$d(m, n) = \frac{1}{r+1} \in \mathbb{R}^+$$

if  $m \neq n$ . For symmetry, note that

$$d(m, n) = d(n, m)$$

since

$$p^r \parallel m - n. \Rightarrow p^r \parallel n - m.$$

For the triangle inequality we would like to show

$$d(m, n) \leq d(m, l) + d(l, n)$$

Where  $d(m, n) = \frac{1}{r+1}$ ,  $d(m, l) = \frac{1}{q+1}$  where  $q$  is such that  $p^q \parallel m - l$ , and lastly  $d(l, n) = \frac{1}{s+1}$  where  $s$  is such that  $p^s \parallel l - n$ . I claim that

$$\frac{1}{r+1} \leq \frac{1}{q+1} + \frac{1}{s+1}.$$

It is clear to see that since  $p^q \mid m - l$  and  $p^s \mid l - n$  then

$$p^{\min\{q,s\}} \mid (m - l) + (l - n) = m - n.$$

This implies that  $r \geq \min\{q, s\}$  as  $p^r \parallel m - n$ . Thus we have

$$\begin{aligned} \frac{1}{r+1} &\leq \frac{1}{\min\{q, s\} + 1} \\ &\leq \frac{1}{q+1} + \frac{1}{s+1} \end{aligned}$$

and the inequality is proven, as claimed. We must next determine if the set

$$2\mathbb{Z} := \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$

is closed in  $\mathbb{Z}$  given the induced metric above. I claim  $2\mathbb{Z}$  is not closed. Consider the following limit

$$\lim_{n \rightarrow \infty} 1 + p^n.$$

Since  $d(1 + p^n, 1)$  goes to 0 as  $n \rightarrow \infty$  we have that the limit is equal to 1, but  $1 + p^n \in 2\mathbb{Z}$  thus  $2\mathbb{Z}$  does not contain all of its limit points as  $1 \notin 2\mathbb{Z}$ .  $\square$

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## 27. Compact subset of Hausdorff separated by $x$ in its complement.

Let  $X$  be a Hausdorff topological space and  $A \subset X$  compact. Show for every  $x \in X \setminus A$  there exists disjoint open neighborhoods

$$U \ni x, V \supset A.$$

*Proof.* Let  $x \in X \setminus A$ . Pick an arbitrary  $a \in A$ . Then by the Hausdorff property, there exists open sets  $U_a, V_a \subset X$  such that

$$x \in U_a, a \in V_a$$

with

$$U_a \cap V_a = \emptyset.$$

It is clear to see that

$$A \subset \bigcup_{a \in A} V_a$$

is an open cover. By compactness of  $A$ , there exists a finite subset  $B \subset A$  such that

$$A \subset \bigcup_{a \in B} V_a = V.$$

Then we can take

$$U = \bigcap_{a \in B} U_a,$$

where we clearly have that  $x \in U$  and since for each  $a$ ,  $U_a \cap V_a = \emptyset$ , we have

$$U \cap V = \emptyset$$

as needed. Note  $A \subset X$  is closed so we have shown  $X$  is regular. □

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## 28. No continuous map between $\mathbb{D}^1$ and $S^1$ .

Show there does not exist a continuous map between  $\mathbb{D}^1$  and  $S^1$  such that  $f(z) = z$  for every  $z \in S^1$ .

*Proof.* Suppose towards a contradiction that there exists such a map. Call it  $f$  where

$$f : \mathbb{D}^1 \rightarrow S^1$$

is continuous. As  $\mathbb{D}^1$  is convex, it is contractible. To see this fix  $z_0 \in \mathbb{D}^1$  and let  $z \in \mathbb{D}^1$  be arbitrary then the path containing, then define

$$\gamma : [0, 1] \times \mathbb{D}^1 \rightarrow \mathbb{D}^1$$

via

$$(t, z) \mapsto z_0 + t(z - z_0)$$

and therefore any and closed path

$$\bar{\gamma} : S^1 \rightarrow \mathbb{C}$$

can be shrunk to  $z_0$  by composing it with  $\gamma$  and thus  $\mathbb{D}^1$  contractible. For path-connected, let  $a, b \in \mathbb{D}^1$  and define

$$\gamma := (1 - t)a + tb$$

for  $t \in [0, 1]$ . I claim  $|\gamma(t)| \leq 1$ .

$$\begin{aligned} |\gamma(t)| &= |(1 - t)a + tb| \\ &\leq |(1 - t)a| + |tb| \\ &= (1 - t)|a| + t|a| \\ &\leq 1 \end{aligned}$$

The last line is because  $a, b \in \mathbb{D}^1$ . Additionally, since

$$\pi_1(S^1) = \mathbb{Z}$$

this implies  $S^1$  is not simply connected, contradicting the existence of such an  $f$ , thus not continuous map exists between  $\mathbb{D}^1$  and  $S^1$ .  $\square$

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## 29. Locally compact and Hausdorff implies regular

Let  $X$  be a locally compact topological space. Prove  $X$  is regular.

*Proof.* Let  $X$  be a locally compact Hausdorff topological space. We show  $X$  is regular. Let  $x \in X$  and  $Y \subset X$  closed be given. Take  $K$  to be the compact neighborhood of  $x$  guaranteed to us by locally compactness. As  $Y \cap K$  is compact and  $x \notin Y \cap K$ , there exists disjoint open set  $U, V$  such that

$$x \in U, Y \cap K \subset V.$$

Note that  $x \in \text{int}K$  and  $K$  is closed since it is a compact subset of a Hausdorff space. Define the open set around  $x$  to be

$$\mathcal{U} = U \cap \text{int}K$$

Then  $x \in \mathcal{U}$  and define

$$\mathcal{V} = V \cup (X \setminus K)$$

then we have that  $F \subset \mathcal{V}$  and

$$\mathcal{U} \cap \mathcal{V} = \emptyset,$$

as needed. □

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### 30. Show $S^1$ is connected and compact.

*Proof.* Show  $S^1$  is connected and compact. For compact we will show it is closed and bounded. First note  $\{1\} \subset \mathbb{R}$  is closed in the map

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

defined via

$$(x, y) \mapsto x^2 + y^2.$$

Thus the preimage of  $\{1\}$  under  $f$  which is continuous is closed and is precisely the unit circle, i.e.,

$$f^{-1}(\{1\}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is closed. It is bounded cause it is contained within  $B_2(0)$  and thus compact. For connected, we prove it is path-connected and thus connected. To show path-connected, let  $a, b \in S^1$  be distinct, then one can write

$$a = (\cos x, \sin x), b = (\cos y, \sin y).$$

Then define

$$g : [0, 1] \rightarrow S^1$$

via

$$g(t) := (\cos t(y - x) + x, \sin t(y - x) + x)$$

then clearly we have

$$g(0) = a, g(1) = b$$

and the space  $S^1$  is path-connected and therefore connected. □

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### 31. Disjoint compact subspaces of Hausdorff space can be separated.

Let  $A, B \subset X$  be disjoint compact spaces and suppose  $X$  is Hausdorff.

*Proof.* We know since  $X$  is Hausdorff and  $B$  is compact, for each  $a \in A$  there exists open neighborhoods  $U_a, V_a \subset X$  such that

$$a \in U_a, B \subset V_a.$$

Clearly  $\{U_a\}_{a \in A}$  is a cover for  $A$  which is compact, thus there exists some finite  $F \subset A$  such that

$$A \subset \bigcup_{a \in F} U_a = U$$

then take  $V$  to be

$$V = \bigcap_{a \in F} V_a,$$

then clearly we have  $B \subset V$  and

$$U \cap V = \emptyset,$$

as needed. □

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### 32. Compute pullback of the 2-form

Let

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

be defined via

$$(u, v) \mapsto (u, v, u^2 - v^2).$$

Let  $\omega$  be the 2-form

$$\omega := ydx \wedge dz + ydy \wedge dz.$$

Compute the pullback

$$F^{-1}\omega$$

. We have

$$\begin{aligned} F^{-1}\omega &:= F^{-1}(ydx \wedge dz + xdy \wedge dz) \\ &:= vdu \wedge d(u^2 - v^2) + udv \wedge d(u^2 - v^2) \\ &= vdu \wedge (2udu - 2v dv) + u dv \wedge (2udu - 2v dv) && \text{differentiating} \\ &= 2uv(du \wedge du) - 2v^2(du \wedge dv) + 2u^2(dv \wedge du) - 2uv(dv \wedge dv) && \text{computing wedge product} \\ &= -2v^2(du \wedge dv) + 2u^2(dv \wedge du) && \text{As } du \wedge du = 0 \text{ same for } v \\ &= -2v^2(du \wedge dv) - 2u^2(du \wedge dv) && \text{As } (du \wedge dv) = -(dv \wedge du) \\ &= -2(u^2 + v^2)(du \wedge dv) && \text{simplifying} \end{aligned}$$

as needed for the pullback.

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### 33. Compact hausdorff spaces need be normal

Show if  $X$  is a compact Hausdorff space, then  $X$  is normal.

*Proof.* Let  $X$  be compact Hausdorff, let  $A, B \subset X$  be disjoint and closed. Then they are both compact. And by compactness of  $B$ , and Exercise 27, for each  $a \in A$ , as  $a \in X \setminus B$  there exists disjoint open sets

$$U_a \ni a, V_a \supset B$$

such that

$$U_a \cap V_a = \emptyset.$$

Then clearly  $\{U_a\}_{a \in A}$  is an open cover for  $A$  and since  $A$  is closed subset of compact, by Exercise 2a  $A$  is compact and so there exists some finite subset  $F \subset A$  such that

$$A \subset \bigcup_{a \in F} U_a = U$$

which is open and define the disjoint open set

$$B \subset \bigcap_{a \in F} V_a$$

and since  $U \cap V = \emptyset$  we are done. □

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### 34. Closed subspace of normal space need be normal

Let  $X$  be a normal topological space. Let  $A \subset X$  be closed. Show  $A$  is normal.

*Proof.* Let  $C, D \subset A$  be closed sets. Then they are closed in  $X$  and by normalness of  $X$  we have disjoint open sets

$$U \supset C, V \supset D$$

such that

$$U \cap V = \emptyset.$$

Then take the open sets in  $A$  to be  $U \cap A$  and  $V \cap A$ . Then clearly

$$C \subset U \cap A, D \subset V \cap A$$

as needed. □

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### 35. Continuous function on compact spaced need be uniform

Let

$$f : (X, d_X) \rightarrow (Y, d_Y)$$

be a continuous map defined on a compact metric space  $(X, d_X)$ . Prove  $f$  is uniform.

*Proof.* Let

$$f : (X, d_X) \rightarrow (Y, d_Y)$$

be give and be continuous with  $X$  compact. We show for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Given  $\varepsilon > 0$ , for each  $x \in X$ , there exists a  $\delta_x$  such that

$$f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x)).$$

Now we have that  $\{B_{\frac{\delta_x}{2}}(x)\}_{x \in X}$  is an open cover for  $X$  which is compact thus there exists a finite subset  $A \subset X$  such that

$$X \subset \bigcup_{x \in A} B_{\frac{\delta_x}{2}}(x).$$

Since  $x_1 \in B_{\frac{\delta_{x_0}}{2}}(x_0)$  for some  $x_0 \in A$ , we have that  $x_2 \in B_{\delta_{x_0}}(x_0)$ , this is by the triangle inequality:

$$\begin{aligned} d_X(x_2, x_0) &\leq d_X(x_2, x_1) + d_X(x_1, x_0) \\ &\leq \frac{\delta_{x_0}}{2} + \frac{\delta_{x_0}}{2} \\ &= \delta_{x_0} \end{aligned}$$

Then if  $d_X(x_1, x_2) < \delta$ , we have that

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq d_Y(f(x_1), f(x_0)) + d_Y(f(x_0), f(x_2)) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and we have the desired result, as needed. □

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### 36. Homotopy preserved under composition

Let  $F_0, F_1 : X \rightarrow Y$  and  $G_0, G_1 : Y \rightarrow Z$  be continuous such that  $F_0 \simeq F_1, G_0 \simeq G_1$ . Prove then that  $G_0 \circ F_0 \simeq G_1 \circ F_1$ .

*Proof.* Since  $F_0 \simeq F_1$ , one has that there exists a continuous map

$$H_F : X \times [0, 1] \rightarrow Y$$

such that

$$H_F(x, t) := \begin{cases} F_0(x) & t \in [0, \frac{1}{2}] \\ F_1(x) & t \in (\frac{1}{2}, 1]. \end{cases}$$

Since  $G_0 \simeq G_1$ , there exists a continuous map

$$H_G : Y \rightarrow Z$$

such that

$$H_G(x, t) := \begin{cases} G_0(x) & t \in [0, \frac{1}{2}] \\ G_1(x) & t \in (\frac{1}{2}, 1]. \end{cases}$$

To see that  $G_0 \circ F_0 \simeq G_1 \circ F_1$  we define the map

$$H : X \times [0, 1] \rightarrow Z$$

defined via

$$H(x, t) = \begin{cases} G_0(H_F(x, 2t)) & t \in [0, \frac{1}{2}] \\ H_G(F_1(x), 2t - 1) & t \in (\frac{1}{2}, 1] \end{cases}$$

Then we have that

$$H(x, 0) = G_0(F_0(x)), H(x, 1) = G_1(F_1(x)).$$

and we have the desired homotopy. □

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### 37. Pullback of 1-form

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be defined via

$$(s, t) \mapsto (\sin t, st^2, s^2 - 1)$$

Let  $\omega$  be the 1-form

$$\omega = dx + xdy + y^2dz$$

Compute the pullback  $f^*\omega$ .

*Proof.* I computed  $f^*\omega$  to be (the pull back under  $\omega$ ) as:

$$\begin{aligned} f^*\omega &= f^*(dx + xdy + y^2dz) \\ &= d(\sin t) + \sin t d(st^2) + s^2t^4 d(s^2 - 1) \\ &= \cos t dt + \sin t(t^2 ds + 2st dt) + 2s^3t^4 ds \\ &= \cos t dt + t^2 \sin t ds + 2st \sin t dt + 2s^3t^4 ds \\ &= (\cos t + 2st \sin t) dt + (t^2 \sin t + 2s^3t^4) ds \end{aligned}$$

□

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### 38. Pullback of a 2-form

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

be given the 2-form  $\omega = x dy \wedge dz$  compute the pull back if  $f$  is defined via

$$(\theta, \phi) \mapsto ((2 + \cos \phi) \cos \theta, (2 + \cos \phi) \sin \theta, \sin \phi)$$

.

*Proof.* Then we get

$$\begin{aligned} f^*\omega &= (2 + \cos \phi) \cos \theta d((2 + \cos \phi) \sin \theta) \wedge d(\sin \phi) && \text{definition} \\ &= (2 + \cos \phi) \cos \theta (-\sin \phi \sin \theta d\phi + (2 + \cos \phi) \cos \theta d\theta) \wedge \cos \phi d\phi && \text{differentiating} \\ &= (2 + \cos \phi)^2 \cos^2 \theta \cos \phi d\theta \wedge d\phi && \text{as } d\phi \wedge d\phi = 0 \end{aligned}$$

□

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### 39. Nested sequence of nonempty connected compact subsets of Hausdorff space has nonempty compact, connected intersection.

Let  $X_1 \supset X_2 \supset X_3 \dots$  be a sequence of nonempty connected compact subspaces of  $\mathbb{R}^n$ . Prove that

$$X = \bigcap_{i=1}^{\infty} X_i$$

is nonempty, compact, and connected.

*Proof.* For non empty, suppose towards a contradiction that

$$X = \bigcap_{n=1}^{\infty} X_n = \emptyset.$$

Then define

$$U_n := X_1 \setminus X_n$$

which is open as the  $X_n$  are compact subsets of Hausdorff space and thus closed. Then

$$\bigcup_{n=1}^{\infty} U_n = X_1 \setminus \bigcap_{n=1}^{\infty} X_n$$

but  $\bigcap_{n=1}^{\infty} X_n$  is empty thus

$$\bigcup_{n=1}^{\infty} U_n = X_1$$

which is compact thus there exists some finite subset  $A \subset \mathbb{N}$  such that

$$X_1 = \bigcup_{n \in A} U_n$$

as the  $U_n$  are increasing, (since the  $X_n$  are decreasing) we have that if  $N := \max_{n \in A}$ , then

$$X_1 = \bigcup_{n=1}^{\infty} U_n = U_N$$

Thus  $X_1 = U_N$  which forces

$$X_N = X_1 \setminus U_N = \emptyset$$

contradicting nonemptyness of all the  $X_n$ ,  $n \in \mathbb{N}$ . For connected, suppose towards a contradiction that

$$\bigcap_{i=1}^{\infty} X_i = A \cup B$$

Where  $A, B$  are nonempty, disjoint, and their union is all of  $X$ . Let  $U, V$  be disjoint open sets containing  $A, B$ , respectively this is by separation of compact subspaces of a Hausdorff space, and define

$$F_i := X_i \setminus (U \cup V)$$

it is clear to see that as the  $F_i \subset X_i$ , then we have that

$$\begin{aligned}\bigcap_{i=1}^{\infty} F_i &= \bigcap_{i=1}^{\infty} X_i \setminus (U \cup V) \\ &= \emptyset \setminus (U \cup V)\end{aligned}$$

Forcing  $F_k = \emptyset$  for some  $k \in \{1, 2, \dots\}$ , this implies, for this  $k$ , we have that

$$X_k \subset U \cup V$$

but as  $X_k$  is connected, this implies by the connected Lemma that WLOG  $X_k \subset U$  forcing  $V = \emptyset$  thus  $B = \emptyset$  and there is no separation of the space forcing  $X$  to be connected. Furthermore,  $\bigcup_{i=1}^{\infty} X_i$  is connected. Suppose

$$\bigcup_{i=1}^{\infty} X_i = C \cup D$$

where  $C, D$  are open nonempty and disjoint. Let  $p \in \bigcap_{i=1}^{\infty} X_i$ , then  $p \in X_i$  for all  $i$ . This forces  $p \in C$  or  $p \in D$ , WLOG let us assume  $p \in C$ . Then as  $X_i$  is connected, it must lie entirely within  $C$  or  $D$  but  $p \in X_i$  thus  $X_i \subset C$  for every  $i$  thus  $D = \emptyset$  and there is no separation of the space, forcing  $\bigcup_{i=1}^{\infty} X_i$  to be connected.  $\square$

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## 40. Integral of form

Let  $\theta, \gamma$  be 3-forms on  $\mathbb{S}^7$ . Show that

$$\int_{\mathbb{S}^7} \theta \wedge d\gamma = \int_{\mathbb{S}^7} d\theta \wedge \gamma.$$

*Proof.* We begin by subtracting the LHS from the RHS to show

$$\int_{\mathbb{S}^7} d\theta \wedge \gamma + (-1)\theta \wedge d\gamma = 0.$$

However, we have that

$$\begin{aligned} \int_{\mathbb{S}^7} d\theta \wedge \gamma + (-1)\theta \wedge d\gamma &= \int_{\mathbb{S}^7} d(\theta \wedge \gamma) && \text{definition of wedge product on forms} \\ &= \int_{\partial\mathbb{S}^7} \theta \wedge \gamma && \text{By Stoke's Theorem} \\ &= 0 && \text{as } \partial\mathbb{S}^7 = \emptyset. \end{aligned}$$

Thus we have that

$$\int_{\mathbb{S}^7} \theta \wedge d\gamma = \int_{\mathbb{S}^7} d\theta \wedge \gamma,$$

as needed. □

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**41. Quotient space onto connected space with that the fibers are connected, then  $X$  need be connected.**

Let

$$f : X \rightarrow Y$$

be a quotient map. Suppose  $Y$  is connected and for each  $y \in Y$ ,

$$f^{-1}(\{y\})$$

is connected. Show  $X$  is connected.

*Proof.* Suppose towards a contradiction that

$$X = A \cup B$$

where  $A, B \subset X$  are both open, nonempty, and disjoint. Then  $f(A), f(B)$  are open in  $Y$ , but by connectedness of  $Y$ , these sets cannot be disjoint. So there exists a  $y$  such that

$$y \in f(A) \cap f(B).$$

Then

$$f^{-1}(\{y\}) = (A \cap f^{-1}(\{y\})) \cup (B \cap f^{-1}(\{y\}))$$

where both sets are open in the subspace topology and both nonempty. As  $y \in f(A)$  implies  $f(x) = y$  for some  $x \in A$ . Same for  $B$ . By connectivity of  $f^{-1}(\{y\})$  these sets cannot be disjoint, thus there exists some  $x \in A \cap B$ , contradicting  $A, B$  being disjoint.  $\square$

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## 42. Open subspaces of compact Hausdorff spaces need be locally compact.

Let  $X$  be a compact Hausdorff space. Let  $U \subset X$  be open. We show  $U$  is locally compact.

*Proof.* Let  $x \in U$ , then since  $U$  is open, there exists an open set  $U_0$  such that

$$x \in U_0 \subset U.$$

By nature of the subspace topology, there exists an open set  $V \subset X$  such that

$$U_0 = U \cap V.$$

As  $X$  is compact Hausdorff, it is normal, so there exists a smaller open set  $W \subset X$  such that

$$x \in W \in \overline{W} \subset V \subset X,$$

then we have that

$$x \in W \cap U \subset \overline{W} \cap U \subset U_0 \subset U$$

□

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### 43. Normal iff Closed sets can be covered

Let  $(X, \tau)$  be a topological space. Then  $X$  is normal if and only if for any closed set  $C \subset X$  and  $U \in \tau$  with

$$C \subset U,$$

there exists  $V \in \tau$  such that

$$C \subset V \subset \bar{V} \subset U.$$

*Proof.* Suppose first that  $X$  is normal, Let  $C \subset X$  be closed and let  $U \in \tau$  be such that

$$C \subset U.$$

Then  $X \setminus U$  is closed and thus there exists  $V, W \in \tau$  with

$$C \subset V, X \setminus U \subset W$$

such that

$$V \cap W = \emptyset.$$

Then  $V \subset X \setminus W$  which is closed forcing

$$\bar{V} \subset X \setminus W$$

hence

$$C \subset V \subset \bar{V} \subset X \setminus W \subset U$$

as needed.

On the other hand, suppose every closed set  $C \subset X$  and every  $U \in \tau$  with  $C \subset U$  we can find a smaller open set  $V \in \tau$  such that

$$C \subset V \subset \bar{V} \subset U$$

Let  $C_1, C_2 \subset X$  be closed and disjoint. As they are disjoint, we have that

$$C_1 \subset X \setminus C_2$$

and since  $C_2$  is closed,  $X \setminus C_2$  is open and thus there exists  $V \in \tau$  such that

$$C_1 \subset V \subset \bar{V} \subset X \setminus C_2.$$

Then,  $X \setminus \bar{V}$  is open and contains  $C_2$  and is disjoint from  $V$  and we have normality.  $\square$

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## 44. Continuous iff every net converges

Let

$$f : X \rightarrow Y$$

be a map of topological spaces. Then show  $f$  is continuous if and only if for every net  $(x_\alpha)_{\alpha \in A}$  converging to  $x$ , we have  $(f(x_\alpha))_{\alpha \in A}$  converges to  $f(x)$ .

*Proof.* Suppose first that  $f$  is continuous. Let  $(x_\alpha)_{\alpha \in A}$  converge to  $x$ . Let  $V \subset Y$  be open containing  $f(x)$ . Then

$$x \in f^{-1}(V)$$

which is open by continuity of  $f$ , thus there exists some  $\alpha_0 \in A$  such that

$$x_\alpha \in f^{-1}(V)$$

for all  $\alpha \geq \alpha_0$ . Then we have

$$f(x_\alpha) \in V$$

forcing  $f(x_\alpha) \rightarrow f(x)$ .

Conversely, suppose  $(x_\alpha)_{\alpha \in A}$  converges to  $x$  but  $(f(x_\alpha))_{\alpha \in A}$  does not converge to  $f(x)$ . Then there exists some open  $V \subset Y$  with  $f(x) \in V$  such that there exists an  $\alpha_0 \in A$  such that for all  $\alpha \geq \alpha_0$ ,

$$f(x_\alpha) \notin V,$$

this implies  $x_\alpha \notin f^{-1}(V)$ , but  $x_\alpha \rightarrow x$  thus  $f^{-1}(V)$  cannot be a neighborhood of  $x$ , hence  $f$  fails continuity and we have a contradiction.  $\square$

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## 45. Hausdorff under continuous function

Let

$$f : X \rightarrow Y$$

be 1-1 and continuous. If  $X$  is Hausdorff, this does not imply  $Y$  is Hausdorff. Take  $X$  and topological space with discrete topology,  $f$  the constant map and  $Y$  and infinite topological space with the cofinite topology.

On the other hand, suppose  $Y$  is Hausdorff, we show  $X$  must also be Hausdorff.

*Proof.* Let  $x_1, x_2 \in X$  be distinct. As  $f$  is 1-1,  $f(x_1) \neq f(x_2) \in Y$ . As they're distinct in a Hausdorff space, there exists open sets  $V_1, V_2 \subset Y$  with  $x_1 \in V_1, x_2 \in V_2$  such that

$$V_1 \cap V_2 = \emptyset.$$

Then since  $f$  is continuous,

$$f^{-1}(V_1), f^{-1}(V_2) \subset X$$

are open and since  $f$  and since  $V_1, V_2$  are disjoint,  $f^{-1}(V_1), f^{-1}(V_2)$  are disjoint and contain  $x_1, x_2$  respectively, thus  $X$  is Hausdorff.  $\square$

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## 46. Hausdorff preserved under homeo

Let

$$f : X \rightarrow Y$$

be a homeomorphism. If  $X$  is Hausdorff, then so is  $Y$ .

*Proof.* Let  $y_1, y_2 \in Y$  be distinct. By surjectivity of  $f$ , there exists  $x_1, x_2 \in X$  such that

$$f(x_1) = y_1, f(x_2) = y_2.$$

Since  $f$  is a bijection, we have that

$$f(x_1) \neq f(x_2)$$

and since  $X$  is Hausdorff, there exists open sets

$$U_1, U_2 \subset X$$

with  $x_1 \in U_1, x_2 \in U_2$  such that

$$U_1 \cap U_2 = \emptyset.$$

Then since  $f$  is 1-1,

$$f(U_1) \cap f(U_2) = \emptyset$$

and we have that

$$y_1 \in f(U_1), y_2 \in f(U_2)$$

which are open as  $f$  is an open map.

□

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## 47. Open and connected subset of $\mathbb{R}^2$ implies connected

Let  $U \subset \mathbb{R}^2$  be open and path-connected. We prove  $U$  is path-connected.

*Proof.* Let  $U \subset \mathbb{R}^2$  be connected and open. Let  $a \in U$  and  $A \subset U$  which can join  $a$ . Define

$$C = U \setminus A$$

and therefore  $A \cap C = \emptyset$ . We wish to prove  $A \subset U$  is non-empty closed and open and thus all of  $U$  making  $U$  path connected. For non-empty, clearly  $a \in A$ .

For open let  $x \in A$ , then by openness of  $U$ , there exists an  $\epsilon > 0$  such that

$$B_\epsilon(x) \subset U.$$

As open balls are convex, they are path-connected thus for any point  $y \in B_\epsilon(x)$  there exists a path from  $x$  to  $y$

$$\phi : [x, y] \rightarrow \mathbb{R}^2$$

But as  $A$  is the set of points that joins  $a$ , there is path from  $x$  to  $a$ , namely

$$\psi : [a, x] \rightarrow \mathbb{R}^2$$

Then there exists a path from  $y$  to  $a$  thus  $y \in A$ . And as  $y \in B_\epsilon(x)$ , we conclude that

$$B_\epsilon(x) \subset A$$

forcing  $A \subset U$  to be open.

To show  $A \subset U$  is closed we show  $C = U \setminus A$  is open which is the set of points that cannot be joined with  $a$  via continuous path within  $U$  thus  $C$  is open since every  $y \in B_\epsilon(x)$  cannot be joined for  $\epsilon > 0$  such that

$$B_\epsilon(x) \subset U,$$

otherwise, we can join  $a$  by  $y$  and  $y$  by  $x$  thus  $a$  with  $x$  contradicting the definition of  $C$ .  $\square$

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## 48. Open map of compact space into connected Hausdorff space.

Let  $f : X \rightarrow Y$  be continuous and open. Suppose  $X$  is compact and  $Y$  is connected and Hausdorff. Then  $f$  is onto.

*Proof.* It suffices to show

$$f(X) = Y.$$

As  $Y$  is connected, it suffices to prove  $f(X) \subset Y$  is both open and closed. Since  $X$  is open and  $f$  is an open map we can conclude that  $f(X) \subset Y$  is open. As  $f$  is continuous and  $X$  is compact,  $f(X) \subset Y$  is compact in a Hausdorff space which implies it is closed forcing

$$f(X) = Y$$

as needed. □

## Counters

1. Any space with the indiscrete topology is connected. With the discrete topology everything is disconnected.
2.  $\mathbb{R}_l$  is not connected. Take  $A = (-\infty, 0), B = [0, \infty)$  as separation components.
3.  $\mathbb{R}_K$  is connected hausdorff, but not path connected, not compact, not regulars
4.  $[0, 1]$  is no longer compact in the  $K$ -topology,  $K$  is an infinite subspace in closed unit with no limit point in  $[0, 1]$ .
5. In discrete topology,  $\overline{\mathbb{Q}} = \mathbb{Q}$
6.  $\overline{(0, 1)}$  in  $\mathbb{R}_l$  is  $[0, 1)$ .
7. To show  $\overline{\bigcup A_i}$  is not always contained in  $\bigcup \overline{A_i}$  Consider  $A = \{r_i\}$  is an enumeration of the rationals, then

$$\begin{aligned}
 \overline{\bigcup A_i} &= \overline{\bigcup \{r_i\}} \\
 &= \overline{\mathbb{Q}} \\
 &= \mathbb{R} \\
 &\not\subseteq \bigcup \overline{A_i} \\
 &= \bigcup \{r_i\} \\
 &= \mathbb{Q}.
 \end{aligned}$$

as needed.

8. To show  $\overline{\text{int}(A)}$  is not always contained in  $\text{int}(\overline{A})$  consider  $A = [0, 1)$ .
9. To show  $\text{int}(\overline{A})$  is not always contained in  $\overline{\text{int}(A)}$  consider  $A = \mathbb{Q}$ .
10. indiscrete everything connected, in the discrete not.
11. To show  $\overline{A \setminus B}$  is not contained in  $\overline{A} \setminus \overline{B}$  take  $\mathbb{R}$  and  $\mathbb{Q}$ .
12.  $\mathbb{R}$  with cofinite topology is compact but not Hausdorff.
13. Note that the boundary of subset of a top space,  $\overline{A} \setminus \text{int}(A)$  does not contain all limit points of  $A$ . Take  $A = [0, 1]$  Then the boundary is  $\{0, 1\}$  but the set of limit points of  $A$  is all of  $A$ .
14.  $\overline{\text{int}(A)}$  does not contain all limit points of  $A$  take  $A = \mathbb{Q}$ .

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