# Complex Analysis Comprehesive Exam Solutions - CSULB

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## 1. Compute crazy integral

Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx; \qquad a > 0.$$

*Proof.* Let $\gamma$ be the path along the real axis then circling back counter-clockwise through the upper half-plane, letting the circle get infinitely big. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx &= \operatorname{Re}\left( \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} dx \right) && \operatorname{Re}(e^{ix}) = \cos x \\
&= \operatorname{Re}\left( \int_{\gamma} \frac{e^{ix}}{x^2 + a^2} dx \right) && \text{integral along the complement of } \gamma \text{ vanishes as circle gets bigger} \\
&= \operatorname{Re}\left( 2\pi i \operatorname{Res}\left( \frac{e^{ix}}{x^2 + a^2}; ai \right) \right) && \text{only singularity is as } x = ia. \\
&= \operatorname{Re}\left( 2\pi i \lim_{z \to ia} \frac{e^{ix}}{x + ia} \right) && \text{computing residues} \\
&= \operatorname{Re}\left( 2\pi i \frac{e^{-a}}{2ia} \right) && \text{plugging in} \\
&= \frac{\pi}{ae^a},
\end{aligned}$$

as needed. ∎

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## 2. Show if $\mathbf{Re}(f(z)) = \mathbf{Im}(f(z))$ for all $z$, then $f$ is constant

Suppose $D \subseteq \mathbb{C}$ is connected and

$$f : D \to \mathbb{C}$$

is holomorphic. Then if $\text{Re}(f(z)) = \text{Im}(f(z))$ for all $z \in D$, prove $f$ is constant.

*Proof.* Suppose that

$$\text{Re}(f(z)) = \text{Im}(f(z)).$$

Then one can write

$$f(z) = u(z) + iu(z).$$

Such that

$$u_x = u_y$$

and

$$u_y = -u_x$$

which implies $f'(z) = 0$ for $z \in D$.
**Lemma** If $f'(z) = 0$ for all $z \in D$ a connected domain, then $f(z)$ is constant.

*Proof.* Consider $z_0 \in D$ and some other point $w \in C$, then call this curve $C_w$, then we have that

$$\begin{aligned} f(w) - f(z_0) &= \int_{C_w} f'(z)dz \\ &= \int_{C_w} 0 \cdot dz \\ &= 0 \end{aligned}$$

This implies $f(w) = f(z_0)$ forcing $f$ to be constant as this holds true for any $w \in D$. □

Thus $f$ is constant on $D$ as needed. □

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## 3. Number of zeros of polynomial

Find the number of zeros

$$z^7 - 5z + 3$$

has on

$$\{z \in \mathbb{C} : 1 < |z| < 2\}$$

*Proof.* We split this into cases (i) When $|z| = 1$ and (ii) when $|z| = 2$.
(i) Suppose $|z| = 1$, then define

$$g(z) := z^7 + 3$$

and

$$f(z) := -5z,$$

then if $|z| = 1$, we have

$$\begin{aligned}|g(z)| &\leq 4 \\ &< 5 \\ &= |f(z)|.\end{aligned}$$

Thus there is only 1 zero for $|z| < 1$.
(ii) Next for $|z| = 2$. Define

$$g(z) = -5z + 3$$

and

$$f(z) = z^7.$$

Then if $|z| = 2$, one has

$$\begin{aligned}|g(z)| &\leq 10 \\ &< 128 \\ &= |f(z)|.\end{aligned}$$

Thus for $|z| < 2$ one has 7 zeros, and so on the annulus one has $7 - 1 = 6$ zeros total. ☐

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## 4. If $v = u^2$, then $f$ is constant.

Show that if $f$,

$$f : D \to \mathbb{C}$$

is holomorphic and $D$ is connected with $v(z) = u^2(z)$, then $f$ is constant.

*Proof.* We can write

$$f(x, y) = u(x, y) + iv(x, y)$$

But by assumption we have that

$$v(x, y) = u^2(x, y).$$

By the Cauchy-Riemann equations we have that

$$u_x = v_y$$

implies

$$u_x = 2uu_y.$$

Similarly,

$$u_y = -v_x$$

implies

$$u_y = -2uu_x.$$

Combining these together we see that

$$u_x(1 + 4u^2) = 0,$$

$$u_y(1 + 4u^2) = 0$$

this implies either $u_x, u_y = 0$ or $1 + 4u^2 = 0$, if this is the case then $f$ is either

$$f := \frac{i}{2} - \frac{1}{4}$$

or

$$f := -\frac{i}{2} - \frac{1}{4}$$

in either case we see that $f$ is constant. Suppose $(1 + 4u^2) \neq 0$, then

$$u_x = 0 = u_y$$

forcing $f' = 0$. By theorem 2.10 of Conway, since $D$ is connected, the constant set is both open and closed in $D$ and thus all of $D$ forcing $f$ to be constant. □

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## 5. Compute crazy integral

Compute

$$\int_0^\infty \frac{\ln x}{x^2 + a^2} dx.$$

*Proof.* Consider the substitution $x = a \tan \theta d\theta$, then $dx = a \sec^2 x d\theta$ and we have

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{x^2 + a^2} dx &= \int_0^{\frac{\pi}{2}} \frac{\ln(a \tan \theta)}{a^2(\sec^2 x)} \cdot a \sec^2 x d\theta \\
&= \frac{1}{a} \int_0^{\frac{\pi}{2}} \ln(a \cdot \tan \theta) d\theta \\
&= \frac{1}{a} \left( \int_0^{\frac{\pi}{2}} \ln a d\theta + \int_0^{\frac{\pi}{2}} \ln(\tan \theta) d\theta \right) \\
&= (\frac{1}{a} \ln a)(\theta) \Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi \ln a}{2a}.
\end{aligned}$$

This assumes

$$\int_0^{\frac{\pi}{2}} \ln(\tan \theta) d\theta = 0.$$

To see this, note

$$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx.$$

Then we make the substitution $\theta = \frac{\pi}{2} - \theta$ to obtain

$$\int_0^{\frac{\pi}{2}} \ln(\tan \theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\tan \frac{\pi}{2} - \theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\cot \theta) d\theta.$$

Then if $I$ is the integral, we have that

$$\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} \ln(\tan \theta) d\theta + \int_0^{\frac{\pi}{2}} \ln(\cot \theta) d\theta \\
&= \int_0^{\frac{\pi}{2}} \ln(\tan \theta \cot \theta) d\theta \\
&= \int_0^{\frac{\pi}{2}} \ln(1) d\theta \\
&= 0
\end{aligned}$$

this forces $I = 0$. □

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## 6. If $f' = 0$ on connected domain, then $f$ is constant on $D$.

Let

$$f : D \to \mathbb{C}$$

and $f'(z) = 0$ for every $z \in D$. Show $f$ is constant.

*Proof.* Fix $z_0 \in G$ and let $w_0 = f(z_0)$. Let

$$A := \{z \in D : f(z) = w_0\}.$$

I claim $A = D$ by showing $A \subset D$ is both open and closed. Let $z \in D$ and let $\{z_n\} \subset A$ with

$$\lim_{n \to \infty} z_n = z.$$

Since $f(z_n) = w_0$ for every $n \geq 1$, we have that

$$f(z) = w_0$$

forcing $z \in A$, this shows $A$ is closed. For showing $A$ is open, fix $a \in A$ and let $\epsilon > 0$ be given such that

$$B_\epsilon(a) \subset G.$$

If $z \in B_\epsilon(a)$, define $g(t)$ by

$$g(t) := f(tz + (1-t)a),$$

for $t \in [0, 1]/$ Then by the chain rule we have

$$\lim_{t \to s} \frac{g(t) - g(s)}{t - s} = f'(tz + (1-t)a)(z - a) = 0.$$

That is, $g'(s) = 0$ for every $s \in [0, 1]$. Thus

$$f(z) = g(1) = g(0) = f(a) = w_0,$$

i.e., $B_\epsilon(a) \subset A$, and $A$ is open. □

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## 7. Compute integral of holomorphic function

Let $f$ be entire. Compute

$$F(R) = \int_{\partial B_R(0)} \frac{f(z)}{(z-a)(z-b)} dz, a \neq b \in \mathbb{C}.$$

*Proof.* This splits into three cases:

$$(1)\, R \in (0, \min\{|a|, |b|\})$$
$$(2)\, R \in (\min\{|a|, |b|\}, \max\{|a|, |b|\})$$
$$(3)\, R \in (\max\{|a|, |b|\}, \infty)$$

For (1), the integral becomes zero as the winding number is zero thus

$$F(R) = 0.$$

For (2), WLOG say that

$$|a| < |b|,$$

then we have that the winding number about $a$ is 1 and we obtain the residue at $z = a$:

$$F(R) = 2\pi i \frac{f(a)}{a-b}.$$

Lastly, for (3) we calculate the residues at $z = a$ and $z = b$:

$$\begin{aligned} F(R) &= 2\pi i \left( \frac{f(a)}{a-b} + \frac{f(b)}{b-a} \right) \\ &= 2\pi i \left( \frac{f(a)}{a-b} - \frac{f(b)}{a-b} \right) \\ &= \frac{2\pi i}{a-b}(f(a) - f(b)) \end{aligned}$$

This gives us

$$\lim_{R\to\infty} F(R) = \frac{2\pi i}{a-b}(f(a) - f(b)).$$

On the other hand, by the $ML-$estimate we have that the length of $\partial B_R(0)$ is $2\pi R$ and since $f$ is bounded, this gives us that

$$\begin{aligned} |F(R)| &\leq 2\pi R \frac{\sup|f|}{(R-|a|)(R-|b|)} && \text{by the } ML\text{−estimate.} \\ &= 2\pi \sup|f| \frac{R}{(R-|a|)(R-|b|)} && \text{commutativity of real numbers} \\ &= 0 && \text{as } R \to \infty \end{aligned}$$

This implies, by the zero product property, that

$$f(a) = f(b); \quad \text{for all } a, b \in \mathbb{C}$$

Thus $f$ is constant. ◻

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## 8. Evaluate integral over $|z| = 1$ of $\frac{1}{8z^3-1}$

Evaluate

$$\int_{|z|=1} \frac{1}{8z^3-1} dz,$$

Factoring, we obtain

$$\int_{|z|=1} \frac{1}{(2z-1)(4z^2+2z+1)} dz.$$

Thus by the quadratic formula, we have poles at

$$z = \frac{1}{2}, \frac{-1 \pm 3i}{2}.$$

However the later two points do not lie within out circle thus by Cauchy's integral formula, we define

$$f(z) := \frac{1}{4z^2+2z+1}$$

we have

$$f(\frac{1}{2}) = \frac{1}{2\pi i} \int_{|z|=1}$$

$$\begin{aligned} f(\frac{1}{2}) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{2z-1} dz \\ &= \frac{1}{2\pi i} \frac{1}{2} \int_{|z|=1} \frac{f(z)}{z-\frac{1}{2}} dz \end{aligned}$$

Thus

$$\begin{aligned} \int_{|z|=1} \frac{f(z)}{z-\frac{1}{2}} dz &= 4\pi i f(\frac{1}{2}) \\ &= \frac{4\pi i}{3} \end{aligned}$$

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## 9. Solve for $z$ in $z^4 = -16$

Solve for all $z \in \mathbb{C}$ in

$$z^4 = -16.$$

*Proof.* We have

$$z^4 + 16 = 0$$

or

$$z^4 + 8z^2 + 16 - 8z^2 = 0$$

This factors into

$$(z^2 + 2\sqrt{2}z + 4)(z^2 - 2\sqrt{2}z + 4) = 0.$$

Using the quadratic formula we obtain

$$z = \sqrt{2}(1-i), \sqrt{2}(1+i), \sqrt{2}(-1-i), \sqrt{2}(-1+i)$$

as needed.
Alternatively, one can factor this into

$$(z^2 + 4i)(z^2 - 4i) = 0$$

Thus $z^2 = \pm 4i$. We have then that if $z$=$4i$,

$$z^2 = 4i = 4(\cos x + i \sin x).$$

Thus taking square roots we obtain:

$$\begin{aligned} z &= \pm\sqrt{4i} \\ &= \pm 2(\cos 90 + i \sin 90)^{\frac{1}{2}} \\ &= \pm 2(\cos 45 + i \sin 45) \\ &= \pm 2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \\ &= \pm\sqrt{2}(1+i) \\ &= \sqrt{2}(1+i), \sqrt{2}(-1-i) \end{aligned}$$

Similarly, if $z^2 = -4i$, we have that

$$z^2 = -4i = -4(\cos 270 + i \sin 270).$$

Then we have that

$$\begin{aligned} z &= \pm\sqrt{-4i} \\ &= \pm 2(\cos 270 + i \sin 270)^{\frac{1}{2}} \\ &= \pm 2(\cos 135 + i \sin 135) \\ &= \pm 2(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \\ &= \sqrt{2}(-1+i), \sqrt{2}(1-i) \end{aligned}$$

□

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## 10. Solve $z^5 = -32$

Find all $z \in \mathbb{C}$ such that

$$z^5 = -32.$$

*Proof.* We know if $z = re^{i\theta}$, then every pole will be of the form $z = 2e^{i\theta}$. Namely,

$$z^5 = 2^5 e^{5i\theta} = 32(-1),$$

and since $e^{\pi i - 2\pi i k} = -1$, we can set

$$5i\theta = \pi i - 2\pi i k \Rightarrow \theta = \frac{\pi + 2\pi k}{5}.$$

Hence $k = 0, 1, 2, 3, 4$ gives us values

$$\{\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}\}$$

□

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## 11. Schwarz Lemma

Let $f$ be holomorphic on $|z| \leq R$ with

$$|f(z)| \leq M,$$

on $|z| = R$. Prove that

$$|f(z) - f(0)| \leq \frac{2M|z|}{R}.$$

*Proof.* We start by defining

$$h(z) := \frac{f(z) - f(0)}{z}$$

on $\overline{B_R(0)} \setminus \{0\}$. Then we have that

$$\begin{aligned} |h(z)| &= \left| \frac{f(z) - f(0)}{z} \right| \\ &= \frac{1}{|z|} |f(z) - f(z)| \\ &\leq \frac{2M}{R} \end{aligned}$$

This implies

$$|f(z) - f(0)| \leq \frac{2M|z|}{R}$$

And this holds for all $z \in \overline{B_R(0)}$ by the maximum principle.
Alternatively, one can define

$$g(z) := \frac{1}{2M}(f(Rz) - f(0))$$

and apply the usual Schwarz Lemma to $g(z)$. □

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## 12. Prove $f$ is a polynomial

Let

$$f : \mathbb{C} \to \mathbb{C}$$

be holomorphic. Suppose ther exists some $n \in \mathbb{Z}$ such that

$$\int_{\partial B_1(0)} \frac{f(z)}{(z-a)^n} dz = 0, \qquad a \in B_1(0).$$

Prove $f$ is a polynomial.

*Proof.* By a corollary of Cauchys integral formula, we have that

$$\int_{\partial B_1(0)} \frac{f(z)}{(z-a)^n} dz = 0 = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

This implies

$$f^{(n-1)}(a) = 0$$

forcing $f$ to be a polynomial as $a \in B_1(0)$. Thus extending $f^{(n-1)}(a) = 0$ on all of $\mathbb{C}$, by the identity principle. □

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True False

- TRUE or FALSE: $\sin z$ is bounded on $\mathbb{C}$. False, Use Liouville's Theorem.
- TRUE or FALSE: There is no holomorphic function defined on $B_1(0)$ such that $f'$ ha sa simple pole at 0. True. This would imply existence of some logarithm holomorphic on $B_1(0)$ which it is not.

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## 14. Compute $\int_\gamma \frac{1}{(z-4)(z-1)} dz$ where $\gamma$ centered at 4, radius 1

Compute

$$\int_\gamma \frac{1}{(z-4)(z-1)} dz,$$

where $\gamma$ is the circle centered at 4 of radius 1.

*Proof.* Clearly, our integrand only has a singularity at $z = 4$ inside of $\gamma$ thus we define

$$g(z) := \frac{1}{z-1}$$

Then we have that

$$\int_\gamma \frac{g(z)}{z-4} dz = 2\pi i g(4) = \frac{2\pi i}{3},$$

as needed. □

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# 15. Analytic implies convergent power series

If $f$ is analytic on some ball, say $B_R(a)$, then

$$f(z) = a_n(z-a)^n$$

for $z \in B_R(a)$, is convergent where

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

are the coefficients.

*Proof.* Let $0 < r < R$ be such that

$$\overline{B_r(a)} \subset B_R(a).$$

If we let $\gamma = a + e^{it}$ for $t \in [0, 2\pi]$, then by previous proposition, we have that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw.$$

Where $|w-a| = r$. Note that we can write

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w-a+a-z} \\ &= \frac{1}{(w-a)(1-\frac{z-a}{w-a})} \\ &= \frac{1}{w-a} \sum_{n=0}^{\infty} (\frac{z-a}{w-a})^n \end{aligned}$$

Thus we can write

$$f(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} f(w) dw.$$

Define

$$g_n(w) := \frac{(z-a)^n}{(w-a)^{n+1}} f(w).$$

And since the trace of $\gamma$ is compact, we can define a max via

$$M := \max\{|f(w)| : w \in \gamma([0, 2\pi])\},$$

then we have that

$$\begin{aligned} |g_n(w)| &= \left| \frac{(z-a)^n f(w)}{(w-a)^{n+1}} \right| \\ &\leq \frac{|z-a|^n}{|w-a|^{n+1}} \cdot M \\ &= \frac{|z-a|^n}{r^{n+1}} \cdot M \\ &= \left( \frac{|z-a|}{r} \right)^n \cdot \frac{M}{r} \end{aligned}$$

Defining
$$M_n := \left(\frac{|z-a|}{r}\right)^n \cdot \frac{M}{r},$$
we note that $\sum M_n$ converges as $n \to \infty$ since $\frac{|z-a|}{r} < 1$. Thus by the Weierstrass M-test, the series
$$\sum_{n=0}^{k} g_n(w) = \sum_{n=0}^{k} \frac{f(w)(z-a)^n}{(w-a)^{n+1}} \to \sum_{n=0}^{\infty} \frac{f(w)(z-a)^n}{(w-a)^{n+1}}$$
uniformly. We can then write
$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{f(w)(z-a)^n}{(w-a)^{n+1}} dw \\ &= \frac{1}{2\pi i} \lim_{k\to\infty} \int_\gamma \sum_{n=0}^{k} \frac{f(w)(z-a)^n}{(w-a)^{n+1}} dw \\ &= \frac{1}{2\pi i} \lim_{k\to\infty} \sum_{n=0}^{k} \int_\gamma \frac{f(w)(z-a)^n}{(w-a)^{n+1}} dw \\ &= \sum_{n=0}^{\infty} \left(\frac{(z-a)^n}{2\pi i}\right) \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw\right) (z-a)^n \end{aligned}$$
Where we have that
$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} = \frac{f^{(n)}(a)}{n!} := a_n \qquad \text{by Corollary of Cauchys Theorem}$$
as needed for the coefficients. As the statement does not depend on $r$, this holds for all $z \in B_R(a)$, as needed. ☐

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