Point-Set Topology Select Solutions

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1. Nested closed subsets of compact

Let (X, τ) be a topological space. Let $E_1 \supset E_2 \supset ...$ be closed subsets of X. If X is compact, prove that $\cap_{i \in I} E_i = \emptyset$.

Proof. Let us assume toward a contradiction that

$$\bigcap_{n \in N} E_i = \emptyset$$

Since for each $n \in \mathbb{N}$ we know E_n is closed in X and they are decreasing, we know the $X \setminus E_n$ are increasing. Then we have that for each n,

$$X \subseteq \bigcup_{n \in \mathbb{N}} X \setminus E_n.$$

By compactness of X we have the existence of a finite subet $A \subset \mathbb{N}$ such that

$$\begin{aligned} X &\subseteq \bigcup_{n \in A} X \setminus E_n \\ &= X \setminus E_M, \end{aligned}$$

Where $M \in A$ is the maximal element in terms of set containment. But this implies

$$E_n = \emptyset$$

For every n such that $1 \leq n \leq M$, contradicting non-emptiness. □

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2 Compactness

(a) Closed subspaces of compact topological spaces need be compact.

Proof. Let X be a compact topological space. Suppose $Y \subseteq X$ is closed. We would like to show that Y is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of Y. That is,

$$Y \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

As $Y \subseteq X$ is closed, its complement in X, $X \backslash Y$, is open in X. Moreover, the union of the U_α together with $X \backslash Y$ forms an open cover for X, i.e.,

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha \cup X \backslash Y.$$

Since X is compact, we know there exists a finite subset $B \subseteq A$ such that

$$X \subseteq \bigcup_{i \in B} U_{\alpha_i} \cup X \backslash Y$$

And since $Y \subseteq X$ we have that

$$Y \subseteq \bigcup_{i \in B} U_{\alpha_i}$$

thus we have found a finite subcover of $\{U_\alpha\}$ that cover Y and so Y is compact as needed. □

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(b) Compact subspaces of Hausdorff topological spaces need be closed.

Proof. Let X be a Hausdorff topological space. Suppose $Y \subseteq X$ is compact. We wish to show Y is closed in X. It suffices to show its complement $X \backslash Y$ is open in X. Let $x \in X \backslash Y$, as X is Hausdorff, for each $y \in Y$ there exists open sets $U_y \subseteq X \backslash Y$, $V_y \subseteq Y$ with $x \in U_y, y \in V_y$ such that

$$U_y \cap V_y = \emptyset.$$

Since $V_y \subseteq Y$ is open, $\{V_y \mid y \in Y\}$ is an open cover for Y, that is,

$$Y \subseteq \bigcup_{y \in Y} V_y.$$

By compactness of Y there exists a finite subset $A \subseteq Y$ such that

$$Y \subseteq \bigcup_{y \in A} V_y = V.$$

Then the finite intersection $U = \bigcap_{y \in A} U_y$ is an open neighborhood of x disjoint from V, namely

$$x \in U \subseteq X \backslash Y$$

and thus $X \backslash Y$ is open in X forcing $Y \subseteq X$ to be closed as needed. □

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(c) The image of a compact topological space need be compact under a continuous map.

Proof. Let $f : X \to Y$ be a continuous map between topological spaces. Suppose X is compact. We would like to show that $f(X) \subseteq Y$ is compact. Let $\{V_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for $f(X)$. That is,

$$f(X) \subseteq \bigcup_{\alpha \in A} V_\alpha$$

where $V_\alpha \subseteq Y$ is open for each $\alpha \in A$. As f is continuous, we have that

$$f^{-1}(V_\alpha) \subseteq X$$

is open for each $\alpha \in A$ as well. Then $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ is an open cover for X and since X is compact we have the existence of a finite subset $B \subseteq A$ such that

$$X \subseteq \bigcup_{b \in B} f^{-1}(V_b)$$

then it follows that

$$f(X) \subseteq \bigcup_{b \in B} V_b$$

thus $f(X)$ is compact. □

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3. Regular space equivalence

(a) Define a regular space.

Proof. A topological space X is called *regular* or T_3 if $\forall x \in X$ and any closed subset $F \subset X$ not containing x, $\exists\, U, V \subset X$ open with $x \in U, F \subset V \ni$

$$U \cap V = \emptyset.$$

Namely, we can separate points from closed sets using open sets. □

Assume X is Hausdorff. Then X is regular if and only if for every $x \in X$ with neighborhood $U \subseteq X$ there exists $V \subseteq X$ open with $x \in V$ such that $\overline{V} \subset U$.

Proof. Let X be a Hausdorff topological space. Suppose first that X is also regular. Let $x \in X$ and let $U_x \subseteq X$ be an open neighborhood of x. Then $F = X \backslash U_x$ is closed by definition. Since X is a regular space we can separate x and F with open subsets of X, that is, there exists $V, W \subseteq X$ open with $x \in V, F \subset W$ such that

$$V \cap W = \emptyset.$$

As $x \in U_x$ we have that

$$V \cap F = \emptyset$$

forcing $\overline{V} \subseteq U_x$ as needed.

Next suppose for each $x \in X$ and open neighborhood of x say $U_x \subseteq X$ there exists an open neighborhood $V \subseteq X$ with $x \in V$ such that $\overline{V} \subseteq U_x$. Let $x \in X \backslash F$ where $F \subseteq X$ is closed. We wish to separate these via open subsets of X. As F is closed it follows that $X \backslash F \subseteq X$ is open. Then by our assumption we are guaranteed the existence of an open set $V \subseteq X$ such that

$$\overline{V} \subseteq X \backslash F.$$

Here $x \in V$ is an open neighborhood of x and similarly $X \backslash \overline{V}$ is an open set containing F such that

$$V \cap X \backslash \overline{V} = \emptyset$$

thus X is regular as needed. □

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4. Non-disjoint union of connected is connected

(a) Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of topological spaces. If for each $\alpha \in I$ X_α is connected and

$$\bigcap_{\alpha \in I} X_\alpha \neq \emptyset,$$

then we have that

$$\bigcup_{\alpha \in I} X_\alpha$$

is connected as well.

Note that this fails if we swap the conclusions union with intersection, take $X_1 := S^1, X_2 := \{(x, y) : x = y \in \mathbb{R}\}$, so the circle and the line, their union is connected, their intersection is two disjoint points however.

Proof. First, Recall the Connected lemma:

> Let (X, τ) be a topological space. If $A \cup B$ is a separation of the space and $Y \subset X$ is a connected subspace, then Y lies entirely in A or B.

The proof of this is in Exercise 16. Now assume towards a contradiction that $\bigcup_{\alpha \in I}$ is disconnected. That is, there is a separation. I.e.,

$$\bigcup_{\alpha \in I} X_\alpha = A \cup B$$

Where $A, B \in \tau$ are non-empty and disjoint. Since the intersection of the X_α is non-empty, let $x \in \bigcap_{\alpha \in I} X_\alpha$, then $x \in A$ or in B, let us say $x \in A$. But then B is non-empty thus there exists some $y \in B$ But then $y \in X_\beta$ for some $\beta \in I$ and also $x \in X_\beta$ contradicting the connected lemma as X_β is connected it must lie entirely in A or B and so $\bigcup_{\alpha \in I} X_\alpha$ is connected. $\square$

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(b) The image of a connected space under continuous function need be connected.

Proof. Let $f : X \to Y$ be a continuous map of topological space. Suppose that X is connected. We wish to show that $f(X)$ is connected as well. Let us suppose towards a contradiction that $f(X)$ has a separation, that is,

$$f(X) = A \cup B$$

where $A, B \subsetneq f(X)$ are both nonempty and open. Since f is continuous, $f^{-1}(A), f^{-1}(B) \subset X$ are both open. Moreover, their union is all of X and thus we have formed a separation of X which is connected, a contradiction and so $f(X)$ is connected. $\square$

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(c) Let X, Y be connected. Show $X \times Y$ is connected in the product topology.

Proof. Suppose X, Y are connected topological spaces. We wish to show that with respect to the product topology, $X \times Y$ is connected as well. Fix $(x_0, y_0) \in X \times Y$. As X is connected and homeomorphic to the slice $X \times \{y_0\}$, it follows that $X \times \{y_0\}$ is connected. Similarly, for each $x \in X$ we have that the slice $\{x\} \times Y$ is connected as well. We can now define

$$T_x = (X \times \{y_0\}) \cup (\{x\} \times Y)$$

Then $\bigcup_{x \in X} T_x$ is connected by part (a) as the intersection consists of (x, y_0). As $\bigcup_{x \in X} T_x$ is all of $X \times Y$, we are done. $\square$

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5. Equivalence for closed in metric space

Let $(X, d$ be a metric space with $C \subset X$ and some $p \in X$ a point. Prove C is closed if and only if $C \cap \overline{B_R(p)}$ is closed for any $R > 0$ where

$$\overline{B_R(p)} = \{x \in X | d(x,p) \leq R\}.$$

Proof. First suppose C is closed. Since arbitrary intersections of closed spaces need be closed we have that the intersection

$$C \cap \overline{B_R(p)}$$

is closed as $\overline{B_r(p)}$ is closed by definition.
On the other hand suppose that for some $p \in X$ and any $R > 0$ the intersection

$$C \cap \overline{B_R(p)}$$

is closed. We wish to show that C is closed, i.e., $C = \overline{C}$. Clearly we have that $C \subseteq \overline{C}$ thus we are left to show that $\overline{C} \subseteq C$. So let $x \in \overline{C}$ be a limit point. We must show $x \in C$. As X is a metric space we can put x in an epsilon ball, that is,

$$x \in B_\varepsilon(x) := \{y \in X \mid d(x,y) < \varepsilon\}.$$

As $C \cap \overline{B_R(p)}$ is closed for any R, we can take $R = d(x,p) + \varepsilon$. Then we have that

$$B_\varepsilon(x) \subset B_R(p) \subset \overline{B_R(p)}.$$

As x is a limit point of C, $B_\varepsilon(x) \backslash \{x\}$ intersects C non-trivially thus x is a limit point of $C \cap \overline{B_R(p)}$ hence $x \in C \cap \overline{B_R(p)}$ forcing $x \in C$ as needed and C is closed. $\square$

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6. Hausdorff & second-countable

(a) Define *second-countable*

Proof. A topological space X is said to be *second-countable* if it has a countable basis. □

(b) Define Hausdorff.

Proof. A topological space X is said to be Hausdorff if for any pair of distinct elements $x, y \in X$ we can find open subsets of X say U, V with $x \in U, y \in V$ such that

$$U \cap V = \emptyset.$$

□

(c) Prove or disprove: Every metric space equipped with the metric topology is Hausdorff.

Proof. Let (X, d) be a metric space. Then d induces a topology on X, namely the collection of epsilon balls, that is,

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}.$$

This collection forms a basis for a topology on X. I claim X is Hausdorff. Let $x, y \in X$ be distinct. Then we have that $d(x, y) > 0$ so denote this distance by ε_0. Then we have basis elements $x \in B_{\frac{\varepsilon_0}{2}(x)}, y \in B_{\frac{\varepsilon_0}{2}(y)}$. It suffices to show

$$B_{\frac{\varepsilon_0}{2}(x)} \cap B_{\frac{\varepsilon_0}{2}(y)} = \emptyset$$

Suppose there exists some $z \in B_{\frac{\varepsilon_0}{2}(x)} \cup B_{\frac{\varepsilon_0}{2}(y)}$ then $d(x, z), d(z, y) < \frac{\varepsilon_0}{2}$. And so by the triangle inequality we have

$$\begin{aligned} \varepsilon_0 &= d(x, y) \\ &\leq d(x, z) + d(z, y) \\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} \\ &= \varepsilon_0 \end{aligned}$$

a contradiction forcing the intersection to be empty as needed thus (X, d) is Hausdorff. □

(d) Prove or disprove: Every metric space equipped with the metric topology is *second-countable.*

Proof. Consider $\mathbb{R}$ as a metric space equipped with discrete metric, that is

$$d(x, y) = \left\{ \begin{array}{ll} 1 & x \neq y \\ 0 & x = y \end{array} \right.$$

Here the basis elements consist of singleton sets. $\mathbb{R}$ has an uncountable number of points, the basis (the singletons) is uncountable thus not second-countable. □

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7. The product topology

Consider the Product space $Y = \Pi_{n=1}^{\infty}[0,1]$ with the product topology.

(a) Prove Y is Hausdorff.

Proof. As $[0,1]$ is Hausdorff (It is a subspace of the reals which are Hausdorff and it is easy to check subspaces of Hausdorff under subspace topology need be Hausdorff as well) we verify that given a collection $\{(X_i, \tau_i)\}_{i\in I}$ of topological spaces, if they are all assumed to be Hausdorff, then their product

$$(\Pi_{i\in I} X_i, \tau_X)$$

(for short we denote $\Pi_{i\in I} X_i$ by X) is Hausdorff when τ_X is the product topology. Before proceeding, small lemma:

> Lemma If A, B, C are sets such that $B \cap C = \emptyset$, then $A \times B \cap A \times C = \emptyset$ as well and this extends to uncountably infinite case as well.

Now let $x, y \in X$ be distinct. Then there exists (at least one) some $j \in I$ our indexing set such that the components do not agree here, that is, for $x_j, y_j \in X_j$ we have

$$x_j \neq y_j.$$

As X_j is Hausdorff, then there exists $U_j, V_j \in \tau_j$ with $x_j \in U_j, y_j \in V_j$ such that

$$U_j \cap V_j = \emptyset.$$

Then by the definition of the product topology, for each $i \in I \setminus \{j\}$ we can take

$$U_i = V_i = X_i$$

And then define the neighborhoods of x, y as

$$U = \Pi_{i\in I} U_i, V = \Pi_{i\in I} V_i.$$

and since the U_i, V_i are disjoint at j together with our lemma we have that

$$U \cap V = \emptyset$$

with $x \in U, y \in V$ thus (X, τ_X) is Hausdorff as needed. □

(b) Prove $Y = \Pi_{n\in\mathbb{N}}[0,1]_n$ is separable.

Proof. To show Y is separable, we construct a countable subset that is dense in Y. Consider the following subset of Y,

$$A := \{(a_1, a_2, ...) \in Y | \exists N \in \mathbb{N} \ni a_i \in \mathbb{Q} \cap [0,1], 1 \leq i < N\}$$

I.e., all sequences with finitely many rational coordinates. I claim A is dense in Y, to see this we show every open set of Y intersects A nontrivially. Let $x \in U \in \tau_Y$. If $x \in A$ then we are done as

$x \in U \cap A$. Suppose $x \in Y \setminus A$. As we are in the product topology, our basis elements are of the form

$$\Pi_{n=1}^{\infty} U_n \subset U$$

Where $U_n = [0,1]$ for all but finitely many n. And for those finitely many n we have the proper containment $U_n \subset [0,1]$. If we let $x = (x_1, x_2, ...)$, then for finitely many values, call them i, we have

$$x_i \in U_i \subset [0,1]$$

As $\mathbb{Q}$ is dense in $\mathbb{R}$, we have for these finite many i, there exists some rational $q_i \in \mathbb{Q}$ such that

$$q_i \in U_i$$

Then consider the point $y = (y_1, y_2, ...)$ such that

$$y_i = q_i; i \leq n, y_n = x_n; \forall n > i$$

Then $y \in A$ and since U was arbitrary we have that A is dense in Y. $\square$

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8. Continuity When restricted

(a) Let (X, τ_X) be a topological space. If we can write $X = \bigcup_{n\in\mathbb{N}} W_n$ where for each $n \in \mathbb{N}$

$$f \restriction_{:} W_n \to Y$$

is continuous, then

$$f : X \to Y$$

is continuous.

Proof. Let

$$f : X \to Y$$

be a map of topological spaces with topologies τ_X, τ_Y respectively. Let $V \in \tau_Y$. Note that

$$\begin{aligned} f \restriction_k^{-1} (V) &= f^{-1}(V) \cap W_k \\ &\in \tau_X. \end{aligned}$$

As $f \restriction_n$ is continuous for every n. But then we can write

$$\begin{aligned} f^{-1}(V) &= \bigcup_{k\in\mathbb{N}} f^{-1}(V) \cap W_k \\ &\in \tau_X \end{aligned}$$

as any union of open need be open thus f is continuous. □

(b) Let (X, τ_X) be a topological space. If $X = A \cup B$ where A, B are closed and $f \restriction_A : A \to Y, f \restriction_B : B \to Y$ are continuous. Prove

$$f : X \to Y$$

is continuous.

Proof. Let $V \subset Y$ be closed, then just as before, since A is a subset of X, then for all subsets of Y which V is, we have

$$f^{-1}(V) \cap A = (f|_A)^{-1}(V)$$

Holds for when $x \in A$ and $f(x) \in V$. As $V \subset Y$ is closed and the restrictions are continuous, we have that

$$(f|_A)^{-1}(V) \cap A$$
$$(f|_B)^{-1}(V) \cap B$$

Are both closed in A, B respectively thus both are closed in X as well and we have

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \cap (A \cup B) \\ &= (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B) \end{aligned}$$

Which is a finite union of closed thus $f^{-1}(V) \subset X$ is closed as needed thus f is continuous. □

(c) Assume $X = \bigcup_{k=1}^{\infty} E_k$ where the E_k are all closed in X such that each $E_k \to Y$ is continuous, is $X \to Y$ also continuous?

Proof. False. Consider the map

$$f : \mathbb{Z} \to \mathbb{R}$$

where $\mathbb{Z}$ is endowed wuith cofinite topology and $\mathbb{R}$ has the standard topology. Let

$$\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} \{n\}$$

So our E_k are just singletons of integers, then $f|_{E_k}$ is continuous for each k.

To see this, take $C \subset \mathbb{R}$ closed, then we have

$$f|_{E_k}^{-1}(C) = f^{-1}(C) \cap E_k$$

which is either just a singleton or the empty set both of which are closed in $\mathbb{Z}$ with cofinite topology.

On the other hand, f is not continuous as any open set $(a, b) \subset \mathbb{R}$ has a pull back with infinite complement thus not open in $\mathbb{Z}$ with cofinite topology. □

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9. Product Topology

(a) Define the product topology on the product $X = \Pi_{j=1}^{\infty} X_j$.

Proof. The product topology on X is the product

$$\Pi_{j=1}^{\infty} U_j$$

Where $U_j \subseteq X_j$ are open and

$$U_j = X_j$$

$\forall$ but finitely many j, for finite j,

$$U_j \subsetneq X_j$$

proper subset. More formally, if π_β is projection onto the βth coordinate, then

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ is open in } X_\beta\}$$

Then the topology generated by $\bigcup_{\beta \in J} \mathcal{S}_\beta$ is the product topology. □

(b) Show the projection map $p_i : X_1 \times X_2 \to X_i$ is an open map for $i = 1, 2$.

Proof. For each $- = 1, 2$, let τ_i denote the topology on X_i. Let $U \in \tau_\alpha$, then by the definition of the product topology we can write

$$U = \bigcup_{j \in J} \bigcap_{k=1}^{n_j} p_{i_{k,j}}^{-1} U_{k,j},$$

where J is an arbitrary indexing set, $n_j \in \mathbb{N}$ and $i_{k,j} = 1, 2$. Then for every $i = 1, 2$, define $V_{i,k,j} \in \tau_i$ via

$$V_{i,k,j} = \begin{cases} U_{k,j} & ; i = i_{k,j} \\ X_i & ; i \neq i_{k,j} \end{cases}$$

By the definition of projection we have

$$p_{i_{k,j}}^{-1}(U_{k,j}) = V_{1,k,j} \times V_{2,k,j}.$$

And without any loss of generality we can suppose $i = 1$ and compute

$$\begin{aligned} p_1(U) &= \bigcup_{j \in J} p_1\Big(\bigcap_{k=1}^{n_j} p_{i_{k,j}}^{-1}(U_{k,j})\Big) \\ &= \bigcup_{j \in J} p_1\Big(\bigcap_{k=1}^{n_j} (V_{1,k,j} \times V_{2,k,j})\Big) \\ &= \bigcup_{j \in J} p_1\Big(\bigcap_{k=1}^{n_j} V_{1,k,j} \times \bigcap_{k=1}^{n_j} V_{2,k,j}\Big) \\ &= \bigcup_{j \in J} \bigcap_{k=1}^{n_j} V_{1,k,j} \\ &\in \tau_1 \end{aligned}$$

and thus p_1 is an open map. The same proof works for p_2. □

(c) If Y is Hausdorff and

$$f : X \to Y$$

is continuous, prove the graph

$$\Delta = \{(x, f(x)) | x \in X\}$$

is closed in $X \times Y$.

Proof. We show Δ^c is open instead.

Let $(x, y) \in X \times Y \backslash \Delta$, then $y \neq f(x)$.

As $y, f(x) \in Y$ which is Hausdorff they can be separated via open sets of Y.

That is, $\exists U, V \subset Y$ open with $y \in U, f(x) \in V$ s.t.

$$U \cap V = \emptyset$$

By Munkres Theorem 18.1(4) since f is continuous, $\exists\ W \subseteq X$ with $x \in W$ s.t.

$$f(W) \subseteq V$$

Then $W \times U$ is an open neighborhood of (x, y) disjoint from Δ thus $X \times Y \backslash \Delta = \Delta^c$ is open, and therefore Δ is closed. □

(d) If Y is Hausdorff and

$$f : X \to Y$$

is continuous, prove

$$G : X \to X \times Y$$

defined via

$$G(x) = (x, f(x))$$

is a closed map.

Proof. Let $C \subseteq X$ be closed, we wish to show

$$G(C) \subset X \times Y$$

is closed. Let $(x, y) \in X \times Y \backslash G(C)$, then $y \neq f(x)$ which are both in Y.

Since Y is Hausdorff, $\exists U, V \subseteq Y$ both open with $y \in U, f(x) \in V$ such that

$$U \cap V = \emptyset$$

As f is continuous however, $\exists W \subseteq X$ an open neighborhood of x such that

$$f(W) \subseteq V$$

Then $W \times U$ is an open neighborhood of (x, y) disjoint from $G(C)$, thus $G(C)^c$ is open forcing $G(C)$ to be closed $\therefore G$ is a closed map. □

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10. Homeomorphisms

(a) Prove (0,1) with the subspace topology is homeomorphic to $\mathbb{R}$ with the standard topology.

Proof. Let

$$f : (0,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

be defined via

$$f(x) := \pi x - \frac{\pi}{2}$$

and let

$$g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$$

be defined via

$$g(x) := \tan x$$

Then

$$h : (0,1) \to \mathbb{R}$$

defined via

$$\begin{aligned} h(x) &:= g(f(x)) \\ &= \tan(\pi x - \frac{\pi}{2}) \end{aligned}$$

is the desired homeomorphism with inverse defined via

$$h^{-1}(x) := \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2}$$

By calulus we are done. □

(b) Assume X, Y are metric spaces that are homeomorphic. Prove or give coumterexample: X complete implies Y complete, that is, is completeness preserved under cont?

Proof. Part (a) □

(c) Prove $[a, b) \not\cong (c, d)$.

Proof. First a small lemma (proof left to the interested reader; HINT: First restrict the domain, then restrict the range.)

> Lemma: If
>
> $$f : X \to Y$$
>
> is a continuous map of topological spaces, then for any $x \in X$,
>
> $$\overline{f} : X \setminus \{x\} \to Y \setminus \{f(a)\}$$
>
> is continuous as well. Morveover, if f is a homeomorphism, then $\overline{f}$ is a homeomorphism as well.

Now let us assume $[a, b) \cong (c, d)$. Then there exists a homeomorphism

$$g : [a, b) \to (c, d).$$

By our lemma above

$$\overline{g} : [a, b) \setminus \{a\} \to (c, d) \setminus \{g(a)\}$$

is a homeomorphism as well. I.e.,

$$g : (a, b) \to (c, g(a)) \cup (g(a), d)$$

is a homeomorphism. Note that the domain is still connected while the range space is clearly disconnected and since connectedness is a topological property, this contradicts continuity of $\overline{g}$ and thus $[a, b) \ncong (c, d)$ as needed. One can check that the range space in fact has a separation. $\square$

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11. Topology of finite point-set

Let $X = \{1, 2, 3, 4\}$ be given by the topology $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$
(a) Show any two disjoint closed sets have disjoint open neighborhoods

Proof. By definition, the closed sets in X are taken to be the complements of the given open sets, that is,

$$\mathcal{C} = \{X, \emptyset, \{2, 3, 4\}, \{3, 4\}, \{2, 4\}, \{4\}\}$$

are all of the closed subspaces of X. As 4 is an element of every member of $\mathcal{C}$ except $\emptyset$ and $\emptyset$ is disjoint with every non-empty set, the pairs of disjoint closed subspaces are

$$X, \emptyset$$

$$\{2, 3, 4\}, \emptyset$$

$$\{3, 4\}, \emptyset$$

$$\{2, 4\}, \emptyset$$

$$\{4\}, \emptyset$$

where the neighborhood X contains each closed set except for $\emptyset$ and clearly $\emptyset$ is its own neighborhood which is disjoint from X as needed. □

(b) Show (X, τ) is not T_1

Proof. Consider the elements $1, 2$. The open neighborhoods of 2 are

$$\{1, 2\}, \{1, 2, 3\}.$$

Since both of these contain 1 we cannot find open sets for 1 and 2 that do not contain each other thus (X, τ) is not T_1. □

(c) Let $A = \{1, 2, 3\} \subset X$ be endowed with the subspace topology. Find disjoint closed subsets of A that do not have disjoint neighborhoods.

Proof. As A is endowed with the subspace topology we can write out the topology on A as follows

$$\tau_A = \{\emptyset, A, \{1\}, \{1, 2\}, \{1, 3\}\}.$$

Then the closed subsets of A are given by

$$\mathcal{C}_A = \{A, \emptyset, \{2, 3\}, \{3\}, \{2\}\}.$$

Then $\{2\}, \{3\}$ are disjoint closed sets in A. The neighborhoods of $\{2\}$ are

$$A, \{1, 2\},$$

and the neighborhoods of $\{3\}$ are

$$A, \{1, 3\}.$$

And since $\{1, 2\} \cap \{1, 3\} = \{1\}$ is non-empty, we have found disjoint closed subsets of A with non disjoint neighborhoods. □

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12. Closures connected/path

(a) If $X \subseteq Z$ is a connected subset of a topological space, show that $\overline{X} \subseteq Z$ is connected as well.

Proof. Let $X \subseteq Z$ be a connected subspace of a topological space. Suppose towards a contradiction that $\overline{X}$ is not connected. Then there exists a separation of $\overline{X}$, that is,

$$\overline{X} = A \cup B,$$

$A, B \in \tau_{\overline{X}}$ non-empty and disjoint. As X is connected, by the connected Lemma we have WLOG that $X = A \cap X$. As B is non-empty, it contains some b, namely b is a limit point of X and thus B is an open set containing a limit point of X thus it must intersect $X \subset A$ non-trivially contradicting

$$A \cap B = \emptyset.$$

Thus $\overline{X}$ is connected. □

(b) Show (a) fails for path-connected subspaces.

Proof. Take the topologist's sin curve. That is, the function

$$f : \mathbb{R}^+ \to [-1, 1]$$

which is defined via

$$x \mapsto \sin(\frac{1}{x}).$$

Then $\{(x, y) \mid y = \sin(\frac{1}{x})\}$ is path-connected, however the closure given via

$$\{(x, y) \in \mathbb{R}^2 \mid y = \sin(\frac{1}{x})\} \cup \{0\} \times [-1, 1]$$

is not path-connected. □

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13. Cofinite Topology

Let τ be the finite complement topology on $\mathbb{R}$. That is, $U \subseteq \mathbb{R}$ is open if and only if U is empty or $\mathbb{R}\backslash U$ if finite.
(a) Is $(\mathbb{R}, \tau)$ Hausdorff?

Proof. I claim $(\mathbb{R}, \tau)$ is not Hausdorff. Let $x, y \in \mathbb{R}$ and $U \in \tau$ be a neighborhood of x. Then $\mathbb{R}\backslash U$ is finite so we can write

$$\mathbb{R}\backslash U = \{p_1, ..., p_n\}.$$

Similarly we can let $V \in \tau$ be a neighborhood of y and write

$$\mathbb{R}\backslash V = \{q_1, ..., q_m\}.$$

We would like for U, V to have a non-empty intersection. As $\mathbb{R}$ is infinite, we can find a $z \in \mathbb{R}$ such that $z \neq p_i$ for $1 \leq i \leq n$ and $z \neq q_j$ for $1 \leq j \leq m$. This would imply $z \in U \cap V$ and so $(\mathbb{R}, \tau)$ is not Hausdorff. □

(b) Is $(\mathbb{R}, \tau)$ compact?

Proof. I claim $(\mathbb{R}, \tau)$ is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for $\mathbb{R}$, that is

$$\mathbb{R} \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

For each $\alpha \in A$ we know $\mathbb{R}\backslash U_\alpha$ is finite, in particular we can take some $\beta \in A$ and so $\mathbb{R}\backslash U_\beta$ is finite. Then for each $x \in \mathbb{R}\backslash U_\beta$ let $U_x \in \tau$ be the neighborhood containing x. Then it follows that

$$\mathbb{R} \subseteq U_\beta \cup \{U_x \mid x \in \mathbb{R}\backslash U_\beta\}$$

is a finite sub-cover of our arbitrary covering thus $(\mathbb{R}, \tau)$ is Hausdorff as needed. □

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14. Sequence of closed

Let $F_1, F_2, ...$ be a sequence of closed subsets of a topological space X. Suppose that for each $x \in X$ we can find a neighborhood of x say N_x such that $N_x \cap F_j \neq \emptyset$ for onlY finitely many j values. Prove $\bigcup_{j=1}^{\infty} F_j$ is closed.

Proof. We will show the complement with respect to the entire space is open. That is, we show

$$X \backslash \bigcup_{j=1}^{\infty} F_j$$

Let $x_0 \in X \backslash \bigcup_{j=1}^{\infty} F_j$ be arbitrary. We must find a neighborhood of x that is disjoint from $\bigcup_{j=1}^{\infty} F_j$. As $x_0 \in X$ we are guaranteed the existence of neighborhood N_{x_0} such that

$$N_{x_0} \cap F_j \neq \emptyset$$

for finitely many j values. That is, there exists a finite set J such that

$$N_{x_0} \bigcap_{j \in J} F_j \neq \emptyset.$$

Then I claim the open neighborhood of x_0 that is disjoint from $\bigcup_{j=1}^{\infty} F_j$ is given by the intersection

$$N_{x_0} \bigcap_{j \in J} X \backslash F_j.$$

Clearly we have that $x_0 \in N_{x_0} \bigcap_{j \in J} X \backslash F_j$ and is clearly disjoint from $\bigcup_{j=1}^{\infty} F_j$. Moreover, as τ_X is closed under finite intersecions we have that $N_{x_0} \bigcap_{j \in J} X \backslash F_j \in \tau_X$ as needed for our neighborhood of x_0 and so $X \backslash \bigcup_{j=1}^{\infty} F_j$ is open thus $\bigcup_{j=1}^{\infty} F_j$ is closed. □

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15 (Incomplete)

Let (X, τ) be a topological space and let $\mathcal{C}$ be the collection of closed sets. A *filter* on $\mathcal{C}$ is a collection $\mathcal{F}$ of sets from $\mathcal{C}$ such that (1) $\emptyset \notin \mathcal{F}$, (2) If $C_1, C_2 \in \mathcal{F}$, then $C_1 \cap C_2 \in \mathcal{F}$, (3) If $C_1 \subset C_2$ with $C_1 \in \mathcal{F}$ and $C_2 \in \mathcal{C}$, then $C_2 \in \mathcal{F}$. Show that if each filter on $\mathcal{C}$ has non-empty intersection, then (X, τ) is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for X. That is,

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha,$$

where $U_\alpha \in \tau$ for each α. Then by definition we have for each $\alpha \in A$ that $X \backslash U_\alpha \in \mathcal{C}$. Let $\mathcal{B} = \{B \in \tau \mid B \subseteq \bigcup_{i \in I} U_{\alpha_i}\}$ for some finite $I \subset A$. Then $X \backslash B \in \mathcal{C}$ for each $B \in \mathcal{B}$. I claim that

$$\mathcal{F}_\beta = \{X \backslash B \mid B \in \mathcal{B}\}$$

defines a filter on $\mathcal{C}$. For (1), let us suppose $\emptyset \in \mathcal{F}_\beta$, then $\emptyset = X \backslash B$ for some $B \in \mathcal{B}$ which implies $B = X$ and since $B \in \mathcal{B}$, we have that $X \subseteq \bigcup_{i \in I} U_{\alpha_i}$. □

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16. Connected Lemma

Let (X, τ) be a a topological space and $Y \subset X$ a connected subspace endowed with the subspace topology. If $A \cup B$ forms a separation of X, then $Y \subset A$ or $Y \subset B$.

Proof. As $A \cup B$ forms a separation, we have that

$$X = A \cup B$$

where $A, B \in \tau$ are both non-empty and disjoint as a pair. Let us suppose towards a contradiction that there exists $a, b \in Y$ such that $a \in A$ and $b \in B$. I claim then that

$$\begin{aligned} Y \cap X &= Y \cap (A \cup B) \\ &= (Y \cap A) \cup (Y \cap B) \end{aligned}$$

forms a separation of Y. Since Y is endowed with the subspace topology and $A, B \in \tau$ we have that $Y \cap A, Y \cap B \in \tau_Y$. That is, they are both open in Y. If they were not disjoint then there would exists some α such that

$$\alpha \in (Y \cap A) \cap (Y \cap B)$$

contradicting A, B being disjoint as a pair and thus $Y \cap A, Y \cap B$ are disjoint as well. Lastly we know by existence of a, b that they are non-empty thus together they form a separation of Y which is connected a contradiction thus Y must lie entirely within A or B. $\square$

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17. Arbitrary collection of topologies on a set

(a) Let X be set and $\{\tau_\alpha\}_{\alpha\in A}$ be a collection of topologies on X. Prove $\bigcap_{\alpha\in A}\tau_\alpha$ is a topology on X.

Proof. Suppose for each $\alpha \in A$ we have that (X, τ_α) is a topological space, we must show $(X, \bigcap_{\alpha\in A}\tau_\alpha)$ is also a topological space. First we check $\emptyset, X \in \bigcap_{\alpha\in A}\tau_\alpha$. Since for each $\alpha \in A$, τ_α is a topology on X, we have for every $\alpha \in A$ that $\emptyset, X \in \tau_\alpha$ and thus

$$\emptyset, X \in \bigcap_{\alpha\in A}\tau_\alpha$$

as needed. Next suppose that

$$U_1, U_2, ..., U_n \in \bigcap_{\alpha\in A}\tau_\alpha.$$

Then for every $\alpha \in A$ we have

$$U_1, U_2, ..., U_n \in \tau_\alpha.$$

As for each $\alpha \in A$, τ_α is a topology on X, by the closure property we get

$$\bigcap_{i=1}^{n} U_i \in \bigcap_{\alpha\in A}\tau_\alpha.$$

Lastly we must check arbitrary unions are closed. That is if for every $\beta \in B$ some indexing set let us suppose

$$U_\beta \in \bigcap_{\alpha\in A}\tau_\alpha.$$

Then for every $\alpha \in A$

$$U_\beta \in \tau_\alpha$$

which are each a topology as noted before thus by closure property of arbitrary unions we get (for every $\alpha \in A$, that is.)

$$\bigcup_{\beta\in B} U_\beta \in \tau_\alpha,$$

which gives us

$$\bigcup_{\beta\in B} U_\beta \in \bigcap_{\alpha\in A}\tau_\alpha,$$

as needed making $(X, \bigcap_{\alpha\in A}\tau_\alpha)$ a topological space. □

(b) Given an example to show $\bigcup_{\alpha\in A}\tau_\alpha$ is not necessarily a topology given τ_α is a topology for each $\alpha \in A$.

Proof. Let our set X be given as the following three point set

$$X = \{a, b, c\}.$$

Consider the following two topologies on X,

$$\tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{b\}\}.$$

Then their union is given by

$$\tau_1 \cup \tau_2 = \{\emptyset, X, \{a\}, \{b\}\}$$

Which is not closed under even finite unions as

$$\{a\} \cup \{b\} \notin \tau_1 \cup \tau_2$$

Thus unions of topologies need not be a topology. □

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18. Intervals in $\mathbb{R}$ are connected

Prove intervals in $\mathbb{R}$ are connected.

Proof. Let $I \subset \mathbb{R}$ be an interval. To show I is connected, we assume towards a contradiction that I is disconnected. That is, there is a separation of the interval

$$I = A \cup B$$

Where $A, B \in \tau_I$ (the topology on I when given the subspace topology inherited from $\mathbb{R}_{\text{standard}}$), non-empty and disjoint. Thus we are guaranteed existence of $a \in A$ and $b \in B$ such that

$$a \notin B, b \notin A.$$

let $I_0 = [a, b]$. Note that $I_0 \subseteq I$. Then we can define

$$A_0 = A \cap I_0, B_0 = B \cap I_0.$$

Then $A_0 \cup B_0$ forms a separtion of I_0. To see this we already know they are non-empty by the existence of a, b. If they were not disjoint then there exists some $\alpha \in A \cap I_0 \cap B$ contradicting A, B being disjoint as they form a separation of I. Lastly since $A, B \in \tau_I$ we have that (in the subspace topology) $A \cap I_0, B \cap I_0 \in \tau_{I_0}$. Are both open. Note that $A_0 \subset \mathbb{R}$ is non-empty thus in inherits the least upper bound porperty so we can define

$$c := \sup(A_0).$$

However A_0 is closed because B_0 is open thus $c \in \overline{A_0}$. and so $c \notin B_0$. As c is the supremum of A_0, for any $x \in I_0$ with $c < x$ we have that $x \notin A_0$ thus we get

$$(x, b] \subset B_0.$$

But then c (Keep in mind that $c \in I_0$) becomes a limit point of B_0 forcing $c \in \overline{B_0}$ and since $A_0 \cup B_0$ form a separtion of I_0, $c \notin A_0$ contradicting

$$c \in I_0 = A_0 \cup B_0.$$

thus I is connected as there is no separation. □

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19. Subspace of separable is separable

If X is a separable metric space, then so is any subspace Y.

Proof. Let (X, τ) be a topological space. Assume that X is separable, then we show for any subspace $Y \subseteq X$ that Y is separable. If $Y = X$ or $\emptyset$ we are done so we suppose $\emptyset \neq Y \subsetneq X$. We must construct a countable dense subset for Y. Let $A \subset X$ be a countable dense subset. Then we can write

$$A = \{a_1, a_2, ...\}$$

such that

$$\overline{A} = X.$$

That is, for every $x \in X$ and $U_x \in \tau$ containing x we have

$$U_x \setminus \{x\} \cap A \neq \emptyset.$$

If we take $y \in Y$, then for any given $\varepsilon > 0$ we have that

$$B_\varepsilon(y) \cap A \neq \emptyset.$$

As the intersection is non-empty let $x_k \in B_\varepsilon(y) \cap A$. I.e.,

$$x_k \in \{B_\varepsilon(y) \cap A : \varepsilon \in \mathbb{R}^+\}.$$

Thus

$$B_\varepsilon(x_k) \cap Y \neq \emptyset.$$

Then we can take

$$\mathcal{B} = \{(k, \varepsilon) : B_\varepsilon(x_k) \cap Y \neq \emptyset\},$$

which is non-empty. So for each (k, ε) take $y_{k,\varepsilon} \in B_\varepsilon(x_k) \cap Y \neq \emptyset$ and let

$$Z = \{y_{k,\varepsilon} : (k, \varepsilon) \in \mathcal{B}\}.$$

And so we have that $Z \subset Y$ is countable since the elements are pulled from elements who are in A which is countable. We must show Z is dense in Y. That is, we must show

$$\overline{Z} = Y.$$

Let $y \in Y$ and $r > 0$ and choose ε such that

$$\varepsilon \leq \frac{r}{2}.$$

Then we can always find a $k \in \mathbb{N}$ such that

$$x_k \in B_\varepsilon(y).$$

Then $(k, \varepsilon) \in \mathcal{B}$ and by the triangle inequality,

$$\begin{aligned} d(y, y_{k,\varepsilon}) &\leq d(y, x_k) + d(x_k, y_{k,\varepsilon}) \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &\leq r. \end{aligned}$$

Thus $y_{k,\varepsilon} \in B_r(y)$ and thus $y \in \overline{Z}$ making Z dense in Y so Y is separable. □

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20. Hausdorff Diagonal

Let (X, τ) be a topological space. Then X is Hausdorff if and only if $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proof. First let us assume that Δ is closed in $X \times X$. We wish to show X is Hausdorff so let $x, y \in X$ be arbitrary. As $\Delta \subset X \times X$ is closed, by definition we know that $\Delta^c \in \tau$. Then there exists a basis element of the form

$$U \times V \quad ; U, V \in \tau,$$

such that

$$(x, y) \in U \times V \subset \Delta^c.$$

And so we have that

$$(U \times V) \cap \Delta = \emptyset$$

which gives us $x \in U, y \in V$. Lastly, I claim that

$$U \cap V = \emptyset.$$

If not, then there exists some $z \in U \cap V$ forcing

$$(z, z) \in U \times V.$$

Moreover, $(z, z) \in \Delta$ by definition, contradicting disjointess of $U \times V$ and Δ and so X is Hausdorff. On the other hand, let us assume that X is Hausdorff and we wish to show that $\Delta \subset X \times X$ is closed. We show this by showing Δ^c is open. Let $x, y \in \Delta^c$. As X is Hausdorff we are guaranteed the existence of $U_x, U_y \in \tau$ such that

$$U_x \cap U_y = \emptyset.$$

I claim that

$$(U_x \times U_y) \cap \Delta.$$

Let $(a, b) \in (U_x \times U_y) \cap \Delta$ then $a = b \in U_x \cap U_y$ a contradiction and thus $\Delta^c \in \tau$ and so $\Delta \subset X \times X$ is closed. □

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21. Equivalence of continuity in metric space

A function f is continuous using open sets if and only if it is continuous in the $\varepsilon - \delta$ sense. Let τ_X, τ_Y denote topologies in X and Y respectively.

Proof. Let X, Y be metric spaces and

$$f : X \to Y$$

a map between them. First we will assume f is open set continuous. That is, if $V \in \tau_Y$ then $f^{-1}(V) \in \tau_X$. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then $f(x_0) \in Y$ and we have that

$$(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \in \tau_Y.$$

Then since f is continuous we get

$$x_0 \in f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)) \in \tau_X.$$

So we can find a basis element containing x_0 fully contained in $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$. I.e., there exists $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon)).$$

But then we have

$$f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

Thus for any $\varepsilon > 0$ we can always find a $\delta > 0$ such that if

$$|x - x_0| < \delta$$

then

$$|f(x) - f(x_0)| < \varepsilon$$

making f continuous in the $\varepsilon - \delta$ senses.

On the other hand suppose that f is continuous in the $\varepsilon - \delta$ sense and let

$$f : X \to Y$$

be our map. Let $V \in \tau_Y$. We wish to show $f^{-1}(V) \in \tau_X$. Let $x_0 \in f^{-1}(V)$, then $f(x_0) \in V \in \tau_Y$ and so there exists an $\varepsilon > 0$ such that

$$f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset V.$$

And since f is continuous in $\varepsilon - \delta$ sense are guaranteed the existence of some $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}(V).$$

This δ ball is the neighborhood of x_0 properly contained in $f^{-1}(V)$ thus $f^{-1}(V) \in \tau_X$ as needed. □

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22. Continuous function over compact space

(a) Show continuous over compact attain a max.

Proof. Let $f : X \to \mathbb{R}$ be continuous. Suppose that X is compact. Since compactness is a topological property, $f(X)$ is compact in $\mathbb{R}$. By Heine-Borel, subsets of $\mathbb{R}$ under the standard topology are compact if and only if they are both closed and bounded. Thus $f(X) \subset \mathbb{R}$ is closed and bounded. Since $f(X)$ is bounded there exists some $M \in \mathbb{R}$ such that for every $x \in X$ we have that

$$|f(x)| \leq M.$$

This tells us the supremum not only exists but is finite thus we can define

$$a := \sup_{x \in X} f(x)$$

Making a a limit point of $f(X)$ which is closed in Y forcing $a \in f(X)$ then by definition we have that

$$f(x) \leq a$$

for every $f(x) \in f(X)$. Hence $f(X)$ attains its max. □

(b) Show continuous over compact is uniform.

Proof. Let

$$f : X \to Y$$

be a map between metric spaces. If f is continuous and X is compact, prove that f is uniformly continuous. I.e., δ is not dependant on each point. Since f is continuous, for each $x \in X$ and any given $\varepsilon > 0$, there is a δ_x such that if

$$d_X(x, y) < \delta_x,$$

then

$$d_Y(f(x), f(y)) < \varepsilon.$$

In other words

$$f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x)). \qquad (*)$$

We now have that $\{B_{\frac{\delta_x}{2}}(x)\}_{x \in X}$ is an open cover for X. As X is compact, we can find a finite subset $A \subset X$ such that

$$X \subseteq \bigcup_{x \in A} B_{\frac{\delta}{2}}(x).$$

Then we can take our δ to be

$$\delta = \min_{x \in A}(\frac{\delta_x}{2})$$

Then we have that $d_X(x, y) < \delta$, then since $x \in B_{\frac{\delta}{2}}(x)$ we have that $y \in B_{\delta_x}(x)$ (keep in mind $x \in A$). Lastly, if $d_X(x, y) < \delta$ I claim $d_Y(f(x), f(y)) < \varepsilon$. Applying $(*)$ we get that

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(z)) + d_Y(f(z), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

as needed. □

Alternate proof:

Proof. Let

$$f : X \to Y$$

be a map between metric spaces. If f is continuous and X is compact, then we show f is uniformly continuous. We say f is continuous in the $\varepsilon - \delta$ at $x \in X$ if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that if

$$d_X(x, y) < \delta,$$

then

$$d_Y(f(x), f(y)) < \varepsilon.$$

I claim that if f is $\frac{\varepsilon}{2} - \delta$ continuous, then f is $\varepsilon - \frac{\delta}{2}$ continuous. To see this, note that for every $x' \in B_{\frac{\delta}{2}}(x)$ and $y \in B_{\frac{\delta}{2}}(x')$ we have that $x', y \in B_\delta(x)$. Thus we can compute

$$\begin{aligned} d_Y(f(x'), f(y)) &\leq d_Y(f(x'), f(x)) + d_Y(f(x), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

And we have proven the claim. Let $\varepsilon > 0$ and $x \in X$. By $\varepsilon - \delta$ continuity, there is some $n \in \mathbb{N}$ such that f is $\frac{\varepsilon}{2} - \frac{1}{n}$ continuous. Then by the claim, f is $\varepsilon - \frac{1}{2n}$ continuous. Moreover, as X is compact, it can be covered by a finite number of these balls so let n_0 be the max n value in the finite collection, then f is $\varepsilon - \frac{1}{2n_0}$ continuous on every neighborhood of X and thus on all of X. □

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23. Countable product of separable

Prove the countable product of separable is separable.

Proof. Let $\{X_n\} n \in \mathbb{N}$ be a collection of separable metric spaces. Let D_n be the associated countable dense subset and fix $x_n \in D_n$. Then for each $m \in \mathbb{N}$, we can define

$$\begin{aligned} E_m &= \{y \in D_n : y_n = x_n; \forall n \geq m\} \\ &= \Pi_{1 \leq n < m} D_k \times \Pi_{n \geq m} \{x_n\}. \end{aligned}$$

Which is clearly countable and thus $\bigcup_m E_m$ is countable as well. I claim that $\bigcup_m E_m$ is dense in $\Pi_n X_n$. Note by the definition of product topology we can find a basis element of the form

$$B = \Pi_{1 \leq n < m} V_n \times \Pi_{n \geq m} X_n.$$

Where $V_n \subset X_n$ open. This is since for all but finitely many, the open sets are the whole space, in the product topology and thus

$$B \cap \bigcup_m E_m \neq \emptyset$$

forcing $\bigcup_m E_m$ to be dense in $\Pi_n X_n$ as needed. □

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24. Closed intervals are compact

Prove for $a < b \in \mathbb{R}$ that $[a, b]$ is compact.

Proof. Let $[a, b]$ for $a < b$ real numbers be an interval. We would like to show it is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover for $[a, b]$ (A some arbitrary indexing set). That is,

$$[a, b] \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Let

$$F := \{x \in [a, b] \mid \exists B \subset A, \text{finite} \ni [a, x] \subseteq \bigcup_{\alpha \in B} U_\alpha\}.$$

Note that $F \neq \emptyset$. This is because $a \in F$ as the empty set is compact. We have that F is a non-empty subset of $\mathbb{R}$ thus it inherits the least-upper-bound property. So we define

$$c := \sup F \in [a, b].$$

I claim that $c = b$. We know $c > a$ because for $\varepsilon > 0$ we there is a neighborhood U_i such that

$$[a, a + \varepsilon] \subset U_i$$

Thus $x \geq a + \varepsilon$ and we know

$$a < c < b.$$

Now take $\beta \in A$ such that $c \in U_\beta$ and choose $\varepsilon > 0$ such that

$$a \leq c - \varepsilon < c < c + \varepsilon \leq b$$

and

$$[c - \varepsilon, c + \varepsilon] \subset U_\beta$$

Since $c - \varepsilon$ is not an upper bound of F there is some c_0 with

$$c - \varepsilon \leq c_0 \leq c$$

such that $c_0 \in F$. which means $[a, c_0]$ has a finite sub-cover from our original cover. I.e.,

$$[a, c_0] \subset \bigcup_{\alpha \in B} U_\alpha$$

which implies

$$[a, c + \varepsilon] \subset \bigcup_{\alpha \in B} U_\alpha \cup U_\beta$$

Forcing $c + \varepsilon \in F$ contradicting the fact that c is the upper bound as $c < c + \varepsilon$. Thus $c = \sup F = b$. Lastly, we show $b \in F$. To see this, note for any $\varepsilon > 0$ we know that there exists some $\gamma \in A$ such that

$$[b - \varepsilon, b] \subset U_\gamma.$$

This gives us the existence of some $c_0 \in [b - \varepsilon, b]$ such that $c_0 \in F$. Then we can write

$$\begin{aligned} [a, b] &= [a, c_0] \cup [b - \varepsilon, b] \\ &\subseteq \bigcup_{\alpha \in B} U_\alpha \cup U_\gamma. \end{aligned}$$

Thus $b \in F$ and we have found a finite sub-cover for $[a, b]$ as needed. □

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25. Successor Topology

Let $X = \mathbb{N}$ and equip it with the following topology

$$\tau = \{U \subset \mathbb{N} : (2n-1) \in U \Rightarrow 2n \in U\}.$$

Prove that (X, τ) is locally compact but not compact.

Proof. For locally compact, let $x \in \mathbb{N}$. Then x is even or odd. If x is even, then take U to be

$$U := \{x, x+1, x+2\}$$

and take our compact superset of U to be

$$K := \{x, x+1, x+2, x+3\}$$

Then U is open and we have

$$x \in U \subset K$$

and K is compact as it is a finite point set. Next suppose x is odd, then take U to be

$$U := \{x, x+1\}$$

and our compact set K to be

$$K := \{x, x+1, x+2\}$$

Then we have that

$$x \in U \subset K$$

and (X, τ) is locally compact.
For showing it is not compact, consider the open covering

$$\bigcup_{n \in \mathbb{N}} \{2n-1, 2n\}$$

I claim this has no finite subcovering. If there was, say there exited some finite subset of $\mathbb{N}$ call it A such that

$$\mathcal{A} = \bigcup_{n \in A} \{2n-1, 2n\} \supset \mathbb{N}$$

Then there exists an $N \in \mathbb{N}$ such that

$$N = \max_{n \in A} +1$$

And

$$\{2N-1, 2N\} \notin \mathcal{A}$$

but

$$2N-1, 2N \in \mathbb{N}.$$

Thus (X, τ) is not compact as needed. ☐

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26. Prime induced metric on $\mathbb{Z}$.

Let p be an odd prime. And define $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as $d(m,n) = 0$ if $m = n$ and

$$d(m,n) = \frac{1}{r+1},$$

where r is the largest positive integer such that

$$p^r \mid m - n$$

and we write

$$p^r || m - n.$$

Show d induces a metric on $\mathbb{Z}$.

Proof. Clearly we have $d(m,n) = 0$ by definition. Furthermoe, as $r \in \mathbb{Z}^+$, we have that

$$d(m,n) = \frac{1}{r+1} \in \mathbb{R}^+$$

if $m \neq n$. For symmetry, note that

$$d(m,n) = d(n,m)$$

since

$$p^r || m - n. \Rightarrow p^r || n - m.$$

For the triangle inequality we would like to show

$$d(m,n) \leq d(m,l) + d(l,n)$$

Where $d(m,n) = \frac{1}{r+1}, d(m,l) = \frac{1}{q+1}$ where q is such that $p^q || m - l$, and lastly $d(l,n) = \frac{1}{s+1}$ where s is such that $p^s || l - n$. I claim that

$$\frac{1}{r+1} \leq \frac{1}{q+1} + \frac{1}{s+1}.$$

It is clear to see that since $p^q \mid m - l$ and $p^s \mid l - n$ then

$$p^{\min\{q,s\}} \mid (m-l) + (l-n) = m - n.$$

This implies that $r \geq \min\{q,s\}$ as $p^r || m - n$. Thus we have

$$\begin{aligned} \frac{1}{r+1} &\leq \frac{1}{\min\{q,s\}+1} \\ &\leq \frac{1}{q+1} + \frac{1}{s+1} \end{aligned}$$

and the inequality is proven, as claimed. We must next determine if the set

$$2\mathbb{Z} := \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$$

is closed in $\mathbb{Z}$ given the induced metric above. I claim $2\mathbb{Z}$ is not closed. Consider the following limit

$$\lim_{n\to\infty} 1 + p^n.$$

Since $d(1 + p^n, 1)$ goes to 0 as $n \to \infty$ we have that the limit is equal to 1, but $1 + p^n \in 2\mathbb{Z}$ thus $2\mathbb{Z}$ does not contain all of its limit points as $1 \notin 2\mathbb{Z}$. ∎

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27. Compact subset of Hausdorff separated by x in its complement.

Let X be a Hausdorff topological space and $A \subset X$ compact. Show for every $x \in X \setminus A$ there exists open neighborhoods

$$U \ni x, V \supset A.$$

Proof. Let $x \in X \setminus A$. Pick an arbitrary $a \in A$. Then by the Hausdorff property, there exists open sets $U_a, V_a \subset X$ such that

$$x \in U_a, a \in V_a$$

with

$$U_a \cap V_a = \emptyset.$$

It is clear to see that

$$A \subset \bigcup_{a \in A} V_a$$

is an open cover. By compactness of A, there exists a finite subset $B \subset A$ such that

$$A \subset \bigcup_{a \in B} V_a = V.$$

Then we can take

$$U = \bigcap_{a \in B} U_a,$$

where we clearly have that $x \in U$ and since for each a, $U_a \cap V_a = \emptyset$, we have

$$U \cap V = \emptyset$$

as needed, □

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28. No continuous map between $\mathbb{D}^1$ and S^1.

Show there does not exist a continuous map between $\mathbb{D}^1$ and S^1 such that $f(z) = z$ for every $z \in S^1$.

Proof. Suppose towards a contradiction that there exists such a map. Call it f where

$$f : \mathbb{D}^1 \to S^1$$

is continuous. As $\mathbb{D}^1$ is convex, it is contractible and thus simply connected, but since

$$\pi_1(S^1) = \mathbb{Z}$$

this implies S^1 is not simply connected, contradicting the existence of such an f, thus not continuous map exists between $\mathbb{D}^1$ and S^1. □

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29. Locally compact and Hausdorff implies regular

Let X be a locally compact topological space. Prove X is regular.

Proof. Let X be a locally compact Hausdorff topological space. We show X is regular. Let $x \in X$ and $Y \subset X$ closed be given. Take K to be the compact neighborhood of x guaranteed to us by locally compactness. As $Y \cap K$ is compact and $x \notin Y \cap K$, there exists disjoint open set U, V such that

$$x \in U, Y \cap K \subset V.$$

Note that $x \in \text{int}K$ and K is closed since it is a compact subset of a Hausdorff space. Define the open set around x to be

$$\mathcal{U} = U \cap \text{int}K$$

Then $x \in \mathcal{U}$ and define

$$\mathcal{V} = V \cup (X \setminus K)$$

then we have that $F \subset \mathcal{V}$ and

$$\mathcal{U} \cap \mathcal{V} = \emptyset,$$

as needed. □

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30. Show S^1 is connected and compact.

Proof. Show S^1 is connected and compact. For compact we will show it is closed and bounded. First note $\{1\} \subset \mathbb{R}$ is closed in the map

$$f : \mathbb{R}^2 \to \mathbb{R}$$

defined via

$$(x, y) \mapsto x^2 + y^2.$$

Thus the preimage of $\{1\}$ under f which is continuous is closed and is precisely the unit circle, i.e,

$$f^{-1}(\{1\}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is closed. It is bounded cause it is contained within $B_2(0)$ and thus compact. For connected, suppose

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^1 = 1\} = A \cup B$$

such that A, B are disjoint and clopen □

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Counters

1. Any space with the indiscrete topology is connected. With the discrete topology everything is disconnected.

2. $\mathbb{R}_l$ is not connected. Take $A = (-\infty, 0), B = [0, \infty)$ as separtion components.

3. $\mathbb{R}_K$ is connected hausdorff, but not path connected, not compact, not regulars

4. $[0, 1]$ is no longer compact in the K-topology, K is an infinite subspace in closed unit with no limit point in $[0, 1]$.

5. In discrete topology, $\overline{\mathbb{Q}} = \mathbb{Q}$

6. $\overline{(0, 1)}$ in $\mathbb{R}_l$ is $[0, 1)$.

7. To show $\overline{\bigcup A_i}$ is not always contained in $\bigcup_i \overline{A_i}$ Consider $A = \{r_i\}$ is an enumeration of the rationals, then

$$\begin{aligned}
\overline{\bigcup A_i} &= \overline{\bigcup \{r_i\}} \\
&= \overline{\mathbb{Q}} \\
&= \mathbb{R} \\
&\not\subseteq \bigcup \overline{A_i} \\
&= \bigcup \{r_i\} \\
&= \mathbb{Q}.
\end{aligned}$$

as needed.

8. To show $\overline{\text{int}(A)}$ is not always contained in $\text{int}(\overline{A})$ consider $A = [0, 1)$.

9. To show $\text{int}(\overline{A})$ is not always contained in $\overline{\text{int}(A)}$ consider $A = \mathbb{Q}$.

10. indiscrete everything connected, in the discrete not.

11. To show $\overline{A \setminus B}$ is not contained in $\overline{A} \setminus \overline{B}$ take $\mathbb{R}$ and $\mathbb{Q}$.

12. $\mathbb{R}$ with cofinite topology is compact but not Hausdorff.

13. Note that the boundary of subset of a top space, $\overline{A} \setminus \text{int}(A)$ does not contain all limit points of A. Take $A = [0, 1]$ Then the boundary is $\{0, 1\}$ but the set of limit points of A is all of A.

14. $\overline{\text{int}(A)}$ does not contain all limit points of A take $A = \mathbb{Q}$.
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