Real Analysis Comprehesive Exam Solutions - CSULB

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1 (Fall 18' Problem 1)

(a) State Fatou's Lemma and the Dominated Convergence Theorem.

Proof. Fatou's Lemma: Let $\{f_n\}$ be a sequence of nonnegative measurable functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then

$$\int f \leq \lim_{n\to\infty} \inf \int f_n.$$

Dominated Convergence Theorem:

Let $\{f_n\}$ be a sequence of measurable functions $\ni f_n(x) \to f(x)$ a.e. x as $n \to \infty$.

If $|f_n(x)| \leq g(x)$, g integrable, then

$$\int |f_n - f| \to 0; \text{as } n \to \infty$$

and thus

$$\int f_n \to \int f; \text{as } n \to \infty$$

as needed. ☐

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(b) Show that Fatou's Lemma implies the DCT.

Proof. Let $\{f_n\}$ be a sequence of measurable functions with $f_n \to f$ a.e. Suppose there exists and integrable function g such that $|f_n| \leq g$, then we also get $|f| \leq g$. And thus by the reverse triangle inequality we have that

$$\begin{aligned} |f_n - f| &\leq |f_n| - |f| \\ &\leq 2g. \end{aligned}$$

We can now define a sequence of non-negative measurable functions $\{h_n\}$

$$h_n(x) := 2g(x) - |f_n(x) - f(x)|$$

where

$$h_n \to 2g$$

as $n \to \infty$. Then applying Fatou's lemma in this case gives us

$$\begin{aligned} \int 2g &\leq \liminf \int 2g - |f_n - f| \\ &= \int 2g - \liminf \int |f_n - f| \end{aligned}$$

Thus we get

$$0 \leq \liminf \int |f_n - f|$$

□

Proof. Let us suppose Fatou's Lemma holds and let $\{f_n\}$ be a sequence of measurable functions such that

$$f_n \to f$$

a.e. as $n \to \infty$ and suppose g is integrable with

$$|f_n(x)| \leq g(x)$$

This give us

$$|f(x)| \leq g(x)$$

Then

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x)| - |f(x)| \\ &\leq 2g(x) \end{aligned}$$

Thus we have a sequence of nonnegative measurable functions

$$h_n(x) := 2g(x) - |f_n(x) - f(x)| \geq 0$$

And since $f_n \to f$ a.e., then $|f(x) - f_n(x)| \to 0$ and by Fatou's lemma

$$\int 2g(x) - 0 \leq \text{lim inf} \int h_n(x)$$

Thus

$$\begin{aligned} \int 2g(x) &\leq \text{lim inf} \int 2g(x) - |f(x) - f_n(x)| \\ &= \int 2g(x) - \text{lim inf} \int |f(x) - f_n(x)| \end{aligned}$$

And by definition of lim inf and lim sup we have

$$\int 2g(x) \leq \int 2g(x) - \text{lim sup} \int |f(x) - f_n(x)|$$

Forcing

$$\text{lim sup} \int |f(x) - f_n(x)| \leq 0(*)$$

and similarly by Fatou's lemma we can obtain

$$\text{lim inf} \int |f(x) - f_n(x)| \geq 0(**)$$

Thus by $(*)$ and $(**)$ we have that

$$\text{lim sup} \int |f(x) - f_n(x)| \leq 0 \leq \text{lim inf} \int |f(x) - f_n(x)|$$

and $\therefore$

$$\text{lim}_{n\to\infty} \int |f(x) - f_n(x)| = 0$$

as needed. □

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2 (Fall 18' Problem 2)

Show the sequence of simple functions $f_i : \mathbb{R} \to \mathbb{R}$

$$f_i(x) = \begin{cases} -1 & i \le x \le i+1 \\ 0 & \text{otherwise} \end{cases}$$

satisfies

$$\liminf_{i\to\infty} \int_{\mathbb{R}} f_i d\lambda < \int_{\mathbb{R}} \liminf_{i\to\infty} f_i d\lambda$$

and why it does not violate Fatou's Lemma.

Proof. To show the inequality holds we can merely compute the RHS and LHS individually. Note that $\forall x \in \mathbb{R}$, the inequality

$$i \le x \le i+1$$

only holds for only finitely many i, so for infinitely many i we have $f_i(x) = 0$ thus

$$\lim_{i\to\infty} f_i(x) = 0$$

And so the RHS is 0, as for the LHS, note

$$\int_{\mathbb{R}} f_i d\lambda = -1$$

Forcing LHS $= -1$ thus

$$\begin{aligned} \liminf_{i\to\infty} \int_{\mathbb{R}} f_i d\lambda &= -1 \\ &< 0 \\ &= \int_{\mathbb{R}} \liminf_{i\to\infty} f_i d\lambda \end{aligned}$$

as needed. Since the f_i are not nonnegative, this does not contradict Fatou. □

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3 (Spring 17' Problem 2)

(a) Suppose we can assume the following:

If each E_n is measurable and $E_n \subseteq E_{n+1}$ $\forall n$, then

$$\lim_{n\to\infty} m(E_n) = m(\bigcup_{n=1}^{\infty} E_n)$$

Then prove that if $m(E_1) < \infty$ and $E_n \supseteq E_{n+1}$ $\forall n$ then

$$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=1}^{\infty} E_n)$$

Proof. Before I proof the problem I will prove what we are supposing, just for fun! So suppose $E_1 \subseteq E_2 \subseteq \ldots$ is an increasing sequence of measurable sets. Then define $G_1 = E_1$ then for every $k \geq 2$ define

$$G_k := E_k \setminus E_{k-1}.$$

Note the G_k are measurable as measurability is preserved under compliments. Moreover the G_k are disjoint and we have $\bigcup_{k\in\mathbb{N}} E_k = \bigcup_{k\in\mathbb{N}} G_k$thus we can compute

$$\begin{aligned}
m(\bigcup_{k\in\mathbb{N}} E_k) &= m(\bigcup_{k\in\mathbb{N}} G_k) && \text{by construction} \\
&= \sum_{k\in\mathbb{N}} m(G_k) && \text{disjointness of the } G_k \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k) && \text{sum of partial sums} \\
&= \lim_{N\to\infty} m(\bigcup_{k=1}^{N} G_k) && \text{disjointness of the } G_k \\
&= \lim_{N\to\infty} m(E_N) && \text{by construction}
\end{aligned}$$

as needed. For the actual problem now, Suppose that $m(E_1) < \infty$, since $E_n \supseteq E_{n+1}$ then by monotonicitcy of the Lebesgue measure we have that

$$m(E_n) < \infty$$

for every n. Thus for every n we can define

$$G_n := E_1 \backslash E_n$$

Then the G_n are increasing. Note that

$$\begin{aligned}
\bigcup_{n=1}^{\infty} E_1 \backslash E_n &= \bigcup_{n=1}^{\infty} (E_1 \cap E_n^c) \\
&= E_1 \cap (\bigcup_{n=1}^{\infty} E_n^c) \\
&= E_1 \cap (\bigcap_{n=1}^{\infty} E_n)^c \\
&= E_1 \backslash (\bigcap_{n=1}^{\infty} E_n)
\end{aligned}$$

Furthermore, as the G_n are increasing, we can apply our lemma from above thus

$$\begin{aligned}
m(E_1) - m(\bigcap_{n=1}^{\infty} E_n) &= m(E_1 \backslash (\bigcap_{n=1}^{\infty} E_n)) \\
&= m(\bigcup_{n=1}^{\infty} E_1 \backslash E_n) \\
&= \lim_{n\to\infty} m(E_1 \backslash E_n) \\
&= \lim_{n\to\infty} [m(E_1) - m(E_n)] \\
&= m(E_1) - \lim_{n\to\infty} m(E_n)
\end{aligned}$$

But $m(E_1)$ is finite thus we can subtract it from both sides leaving us with

$$m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$$

as needed. □

(b) Suppose $K \subseteq \mathbb{R}^d$ is closed and bounded and define

$$E_n = \{x \in \mathbb{R}^d \mid \exists y \in K \text{s.t.} |x - y| < \frac{1}{n}\}$$

and prove that

$$\lim_{n\to\infty} m(E_n) = m(K)$$

Proof. First note that the E_n are an increasing sequence of measurable sets, i.e.,

$$E_n \supseteq E_{n+1}$$

so by our result in (a) we can write the left hand side as

$$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=1}^{\infty} E_n)$$

and so it suffices to verify that

$$m(\bigcap_{n=1}^{\infty} E_n) = m(K)(*)$$

That is we must show

$$\bigcap_{n=1}^{\infty} E_n = K$$

via set containment. Clearly we have $K \subset \bigcap_{n=1}^{\infty} E_n$ as $K \subset E_n$ for every n. Suppose towards a contradiction that $\exists p \in \bigcap_{n=1}^{\infty} E_n \backslash K$ then there exists a sequence $\{y_n\}$ in K such that

$$|y_n - p| < \frac{1}{n}$$

i.e.,

$$y_n \to p$$

makin g p a limit point of K which is a closed and bounded subset of the reals so it must contain all of its limit points, that is $p \in K$ a contradiction forcing $(*)$ to hold true as needed thus

$$\lim_{n\to\infty} m(E_n) = m(K)$$

as desired. □

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4 (Spring 17' Problem 8)

(a) State Fatou's Lemma and the MCT and show Fatou's Lemma implies the MCT.

Proof. Refer to Exercise 1 for Fatou's Lemma. The Monotone Convergence Theorem states that given a sequence of non-negative measurable functions $\{f_n\}$ such that they increase to f, then

$$\lim_{n\to\infty} \int f_n = \int f.$$

□

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(b) Show Fatou implies MCT

Proof. Let us suppose that Fatou's Lemma holds. Let $\{f_n\}$ be a sequence of non-negative measurable functions such that they monotonically increase to f, that is

$$f_n(x) \le f_{n+1}(x)$$

and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

by Fatou's Lemma we have that

$$\int f \le \lim_{n\to\infty} \inf \int f_n . (*)$$

However as the f_n increase to f we have that

$$\int f_n \le \int f$$

which gives

$$\lim_{n\to\infty} \sup \int f_n \le \int f$$

and together with $(*)$ we have that

$$\lim_{n\to\infty} \sup \int f_n = \lim_{n\to\infty} \inf \int f_n$$

thus

$$\lim_{n\to\infty} \int f_n = \int f$$

as needed. □

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5 (Sprint 17' Problem 5)

Let f be a real valued integrable function on $\mathbb{R}^d$. Prove that for $\delta > 0$ we have

$$\int f(\delta x) = \delta^{-d} \int f(x).$$

Proof. Let f be integrable over $\mathbb{R}^d$. By dilation invariance, for any $\delta > 0$ we have that

$$\delta^d m(E) = m(\delta E)$$

thus taking δE to be $\delta \mathbb{R}^d$ we can write

$$\int_{\mathbb{R}^d} f(x) = \delta^d \int_{\mathbb{R}^d} f(\delta x)$$

substituting this in for $\int f(x)$ gives the desired result. □

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6 (Fall 17' Problem 2)

(a) Show that the graph $\Gamma(f) = \{(x, f(x)) \mid x \in [a, b]\}$ of a continuous function f on $[a, b]$ has measure zero without the use of Fubini.

Proof. Let $f[a, b] \to \mathbb{R}$ be continuous. We wish to show that $m(\{(x, f(x) \mid x \in [a, b]\}) = 0$. Let $\epsilon > 0$ be given and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of our interval such that

$$|x_i - x_{i-1}| < \delta$$

for each $i \in \{0, 1, ..., n\}$, then by continuity of f we have that

$$|f(x_i) - f(x_{i-1})| < \epsilon$$

Since $[a, b]$ is compact and f is continuous we can define

$$m_i := \min_{x \in [x_{i-1}, x_i]} f(x), M_i := \max_{x \in [x_{i-1}, x_i]} f(x)$$

and note that

$$\Gamma(f) \subseteq \bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [m_i, M_i]$$

then we have

$$\begin{aligned} m(\Gamma(f)) &\leq m(\bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [m_i, M_i]) \\ &= \sum_{i=1}^{n} (|x_i - x_{i-1}|)(|M_i - m_i|) \\ &< (b - a)\epsilon \end{aligned}$$

where the first inequality holds by monotonicity. And since each $[x_{i-1}, x_i]$ has measure less than $\frac{b-a}{n} = \delta$ we are done. □

(b) Prove that the graph of a continuous function on $\mathbb{R}$ has measure zero.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We wish to show $m(\{(x, f(x) \mid x \in \mathbb{R}\}) = 0$. Let Γ denote the graph. As the map $x \mapsto -x$ preserves the area of rectangles, the measure of Γ is the same as that of $|f(x)|$ thus we may suppose f is nonnegative. Note we can write Γ as

$$\Gamma = \bigcup_{n \in \mathbb{N}} \{(x, f(x)) \mid x \in [n, n+1]\}$$

Then by part (a) taking $a = n, b = n + 1$ together with coutmable subadditivity it follows that $m(\Gamma) = 0$. □

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7 (Spring 17' Problem 9)

(a) *Proof.* □

8

Prove the continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable by showing the following. If $G = \{U \subset \mathbb{R} | g^{-1}(U) \text{is Borel}\}$ (a) open sets are in G. (b) G forms a σ-algebra.

Proof. First we must show open sets are in G. Let $V \subset \mathbb{R}$ be an open subset. Since g is continuous, we have that $g^{-1}(V)$ is open in $\mathbb{R}^n$ and in partiulcar, a Borel set in $\mathbb{R}^n$ thus $V \in G$ as needed.
For the latter, we must verify G satisfies the axioms of a σ-algebra. First, note that $\mathbb{R} \in G$ since $g^{-1}(\mathbb{R}) = \mathbb{R}^n$ is Borel. Suppose that $U_1, U_2, ... \in G$, then for each j, we have that $g^{-1}(U_j)$ is Borel. Thus

$$g^{-1}(\bigcup_{j=1}^{\infty} U_j) = \bigcup_{j=1}^{\infty} g^{-1}(U_j)$$

is Borel as well. Similarly for intersection. Lastly, let us suppose $U \in G$. Then $U \subset \mathbb{R}$ and $g^{-1}(U)$ is Borel. Then clearly we have that

$$\begin{aligned} g^{-1}(U^c) &= g^{-1}(\mathbb{R} \backslash U) \\ &= g^{-1}(\mathbb{R}) \backslash g^{-1}(U) \\ &= \mathbb{R}^n \backslash g^{-1}(U) \\ &= g^{-1}(U)^c \end{aligned}$$

is Borel as well making G a σ-algebra. □

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9 (Fall 18' Problem 5)

Let f, f_k for $k \in \mathbb{N}$ be measurable and finite valued on a measurable set E. Show if $f_k \to f$ almost everywhere on E and if E has finite measure, then $f_k \xrightarrow{m} f$ converges in measure as well

Proof. Fix $\epsilon > 0$ and define

$$E_N = \{x \in E | \exists n > N : |f_n(x) - f(x)| > \epsilon\}.$$

Then it is clear the E_N are decreasing, that is $E_1 \subset E_2 \subset ... \subset E$. Since $\lambda(E) < \infty$, by monotonicity of measure we have that all the E_N have finite measure as well. Furthermore as $f_k \to f$ a.e. on E, we have that

$$\lambda(\bigcap_{N \in \mathbb{N}} E_N) = 0$$

thus the limit of the measures of the E_N is zero as well. So there exists an N such that $\lambda(E_N) < \epsilon$. So for every $n > N$ we have

$$\{x \in E : |f_k(x) - f(x)| > \epsilon\} \subset E_N$$

and so $f_k \xrightarrow{\text{measure}} f$ converges in measure as well. □

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10 (Fall 18' Problem 7)

(a) Suppose f is non-negative and integrable on some measurable function E. For $\alpha > 0$ define $E_\alpha = \{x \in E \mid f(x) > \alpha\}$. Prove that

$$m(E_\alpha) \leq \frac{1}{\alpha}\int_E f dm.$$

Proof. We consider the simple function defined as $g(x) = \alpha\chi_{E_\alpha}(x)$. Then we have that $0 \leq g(x) \leq f(x)$. By Monotonicity we have

$$\begin{aligned} \alpha m(E_\alpha) &= \int_E g dm \\ &\leq \int_E f dm. \end{aligned}$$

As $\alpha > 0$ we divide by α to obtain

$$m(E_\alpha) \leq \frac{1}{\alpha}\int_E f dm$$

as needed. □

(b) Suppose f is non-negative on E and $\int_E f = 0$. Prove $f = 0$ a.e. on E.

Proof. We appeal to the inequality from part (a), take $\alpha = \frac{1}{n}$ where $n \in \mathbb{N}$. Then by (a) we have that

$$\begin{aligned} m(E_{\frac{1}{n}}) &= m(\{x \in E \mid f(x) > \frac{1}{n}\}) \\ &\leq n\int_E f dm \\ &= 0 \end{aligned}$$

thus $E_{\frac{1}{n}}$ has measure 0 and we can compute

$$\begin{aligned} m(E_0) &= m(\{x \in E \mid f(x) > 0\}) \\ &= m(\bigcup_{n\in\mathbb{N}}\{x \in E \mid f(x) > \frac{1}{n}\}) \end{aligned}$$

which is a countable union of measure 0 sets forcing $m(E_0) = 0$ thus $f = 0$ a.e. on E. □

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11 (Fall 20' Problem 1)

Show the exterior meausre of the closed unit interval is 1.

Proof. We will show that $m_*([0,1]) \leq 1$ and $m_*([0,1]) \geq 1$ to conclude the outer measure is 1. First note for $d = 1$ by definition of volume we have the volume of $[0,1]$ is just

$$|[0,1]| = 1.$$

Now to show the first inequality, note that as $[0,1]$ itself is closed, we can take $I_1 = [0,1], I_j = \emptyset$ for $j \geq 2$ as a covering of $[0,1]$, then we have that

$$\begin{aligned} m_*([0,1]) &= \inf\{\sum |I_j|\} \\ &= \inf\{\sum |[0,1]|\} \\ &\leq |[0,1]| \\ &= 1. \end{aligned}$$

Thus $m_*([0,1]) \leq 1$. On the other hand let $\{I_j\}_{j\geq 1}$ be a countable covering of $[0,1]$, that is,

$$[0,1] \subset \bigcup_{j\geq 1} I_j.$$

Then we have that

$$\begin{aligned} \sum_j |I_J| &\geq |\bigcup_j I_j| \\ &\geq |[0,1]| \\ &= 1. \end{aligned}$$

Then $\inf\{\sum_j |I_j|\} \geq 1$ and thus $m_*([0,1]) \geq 1$ as needed forcing $m_*([0,1]) = 1$ as needed.

Next we can show $m_*([0,1] \cap \mathbb{Q}) = 0$. Note since we have

$$[0,1] \cap \mathbb{Q} \subset \mathbb{Q}$$

By monotonicity we have that

$$m_*([0,1] \cap \mathbb{Q}) \leq m_*(\mathbb{Q}).$$

Thus it suffices to show the RHS is equal to 0. First I claim that singleton sets have measure zero. Let $A = \{a\}$. Then for any $\epsilon > 0$ we have that

$$A \subset (a - \epsilon, a + \epsilon).$$

Then by monotonicity we have that

$$\begin{aligned} m_*(A) &\leq m_*((a-\epsilon, a+\epsilon)) \\ &= |(a-\epsilon, a+\epsilon)| \\ &= 2\epsilon \end{aligned}$$

thus A has measure 0. Next I claim any countable subset of $\mathbb{R}$ has measure zero. If we take $B_n := \{b_n\}$ then we can express a countable subset B of $\mathbb{R}$ as

$$B = \bigcup_{n\in\mathbb{N}} B_n.$$

and we can thus compute

$$\begin{aligned} m_*(B) &= m_*(\bigcup_{n\in\mathbb{N}} B_n) \\ &\leq \sum_{n\in\mathbb{N}} m_*(B_n) \\ &= 0, \end{aligned}$$

Forcing $m_*(B) = 0$. It follows that $m_*(\mathbb{Q}) = 0$.

Next we can show that $m_*([0,1]\backslash\mathbb{Q}) = 1$. For $\leq$ note $([0,1] \setminus \mathbb{Q}) \subset [0,1]$ thus by monotnonicity we have

$$\begin{aligned} m_*([0,1] \setminus \mathbb{Q}) &\leq |[0,1]| \\ &= 1. \end{aligned}$$

For the other direction, first note that

$$[0,1] \setminus \mathbb{Q} \cup [0,1] \cap \mathbb{Q} = [0,1].$$

Then we can compute by countable sub-additivity.

$$\begin{aligned} 1 &= m_*([0,1]) \\ &= m_*([0,1] \setminus \mathbb{Q} \cup [0,1] \cap \mathbb{Q}) \\ &\leq m_*([0,1] \setminus \mathbb{Q}) + m_*([0,1] \cap \mathbb{Q}) \\ &= m_*([0,1] \setminus \mathbb{Q}). \end{aligned}$$

giving us $m_*([0,1] \setminus \mathbb{Q}) = 1$. □

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12 (Fall 20' Problem 4)

Let $E = [0,1] \setminus \mathbb{Q}$. Given $\epsilon > 0$ construct a closed set F contained in E such that $m_*(E \setminus F) < \epsilon$.

Proof. Let $\epsilon > 0$ be given. Let $\{q_n\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. We can then define F to be given as

$$F = [0,1] \setminus \bigcup_{i=1}^{\infty} (q_i - \frac{\epsilon}{2^{i+1}}, q_i + \frac{\epsilon}{2^{i+1}})$$

It is clear to see that $F \subset E$. We can now compute

$$\begin{aligned} m(E \setminus F) &= m(\bigcup_{i=1}^{\infty} (q_i - \frac{\epsilon}{2^{i+1}}, q_i + \frac{\epsilon}{2^{i+1}}) \setminus \mathbb{Q}) \\ &= m(\bigcup_{i=1}^{\infty} (q_i - \frac{\epsilon}{2^{i+1}}, q_i + \frac{\epsilon}{2^{i+1}})) - m(\mathbb{Q}) \\ &= m(\bigcup_{i=1}^{\infty} (q_i - \frac{\epsilon}{2^{i+1}}, q_i + \frac{\epsilon}{2^{i+1}})) \\ &= \sum_{i=1}^{\infty} |q_i + \frac{\epsilon}{2^{i+1}} - (q_i - \frac{\epsilon}{2^{i+1}})| \\ &= \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \\ &< \epsilon \end{aligned}$$

as needed. □

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13 (Fall 20' Problem 4)

Let $K \subset \mathbb{R}^d$ be compact and let $\mathcal{O}_n = \{x \in \mathbb{R}^d \mid d(x, K) < \frac{1}{n}\}$. Prove that $\lim_{n\to\infty} m(\mathcal{O}_n) = m(K)$.

Proof. First note that the $\mathcal{O}_n$ are a decreasing sequence of measurable sets. That is, for every n we have

$$\mathcal{O}_n \supseteq \mathcal{O}_{n+1}.$$

Furthermore we have $m(\mathcal{O}_1) < \infty$, by Monotone convergence for decreasing measurable sets we can conclude that

$$\lim_{n\to\infty} m(\mathcal{O}_n) = m(\bigcap_{n=1}^{\infty} \mathcal{O}_n).$$

We are left to show then that

$$m(\bigcap_{n=1}^{\infty} \mathcal{O}_n) = m(K).$$

I.e., we must show

$$\bigcap_{n=1}^{\infty} \mathcal{O}_n = K$$

via set containment. Clearly we have $K \subset \bigcap_{n=1}^{\infty} \mathcal{O}_n$. To show $\bigcap_{n=1}^{\infty} \mathcal{O}_n \subset K$ suppose there exists some $p \in \bigcap_{n=1}^{\infty} \mathcal{O}_n \setminus K$. Then there exists a sequence $\{y_n\}$ in K such that

$$|y_n - p| < \frac{1}{n}.$$

Thus p is a limit point of K which contains its limit points forcing $p \in K$ as needed. □

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14 (Fall 20' Problem 3)

Show if f is Lipshitz and defined on a measure set E, then the measure of $f(E)$ is zero as well.

Proof. Let k be the Lipshitz constant, then for any $x, y \in \mathbb{R}$ we have

$$|f(y) - f(x)| \leq k|y - x|$$

Let E be a measure zero set. Then for any given $\epsilon > 0$ E can be covered by ball of radius r_j,

$$E \subseteq \bigcup_{j=1}^{\infty} B_{r_j}$$

Where $\Sigma_{j=1}^{\infty} m(B_{r_j}) < \epsilon$. By Lipshitz continuity, for each j, $f(B_{r_j})$ is contained in a ball of radius kr_j thus

$$m(f(B_{r_j})) \leq km(B_{r_j})$$

Then we have that

$$\begin{aligned} m(f(E)) &\leq k\sum_{j=1}^{\infty} m(B_{r_j}) \\ &< k\epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $m(f(E) = 0$ as needed. ☐

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15 (Fall 19' Problem 2)

(a) Prove sub-additivity for exterior measure, that is,

$$m_*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m_*(E_k),$$

for arbitrary collection $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{R}$.

Proof. If there is a k such that $m_*(E_k) = \infty$ then we are done. Thus for every k we can assume

$$m_*(E_k) < \infty.$$

Let $\epsilon > 0$ be given. For each k we can find a countable covering of E_k. That is, we are guaranteed to find a collection of closed set $\{I_j^k\}_j$ with $E_k \subseteq \bigcup_j I_j^k$ such that

$$\sum_j |I_j^k| \leq m_*(E_k) + \frac{\epsilon}{2^k}.$$

This gives us

$$\bigcup_k E_k \subseteq \bigcup_{k,j} I_j^k$$

And we can compute

$$\begin{aligned} m_*(\bigcup_k E_k) &\leq \sum_{k,j} |I_j^k| \\ &\leq \sum_k (m_*(E_k) + \frac{\epsilon}{2^k}) \\ &= \sum_k m_*(E_k) + \epsilon \end{aligned}$$

and ϵ was arbitrary thus we are done. □

(b) Prove monotonicity for exterior measure. That is, for $A, B \subset \mathbb{R}$ arbitrary we have if $A \subset B$, then

$$m_*(A) \leq m_*(B).$$

Proof. Let $A, B \in \mathcal{P}(\mathbb{R})$ be arbitrary with $A \subset B$. Then if $\{I_j\}_j$ is any countable cover of B, then it clearly also covers A. Then

$$m_*(A) \leq \sum_j |I_j|,$$

which holds for every countable covering of B thus we can compute

$$\begin{aligned} m_*(A) &\leq \inf\{\sum_j |I_j|\} \\ &= m_*(B). \end{aligned}$$

as needed. □

(c) Let $E \subseteq \mathbb{R}^d$ and $\epsilon > 0$ be given. Show there exists an open set $O \supseteq E$ such that the exterior measures differ by ϵ.

Proof. If $m_*(E) = \infty$, then taking $O = \mathbb{R}$ give the desired result. Suppose then that $m_*(E) < +\infty$ and let $\epsilon > 0$ be given. By definition of the exterior measure we are guaranteed the existence of a countable collection of closed cubes $\{I_j\}_j$ such that

$$E \subseteq \bigcup_j I_j$$

and that

$$\sum_j |I_j| \leq |E| + \frac{\epsilon}{2}.$$

Let I_j^* be a closed cube such that

$$I_j \subseteq \text{int}(I_j^*).$$

and

$$|I_j^*| \leq |I_j| + \frac{\epsilon}{2^{j+1}}.$$

Then we can take our open set to be

$$O = \bigcup_j \text{int}(I_j^*)$$

which is open as arbitrary unions of open are open and it is clear to see $E \subset O$. We can thus compute

$$\begin{aligned} m_*(O) &= \inf \sum_j |\text{int}(I_j^*)| \\ &\leq \sum_j |I_j^*| \\ &\leq \sum_j |I_j| + \frac{\epsilon}{2} \\ &\leq |E| + \epsilon \end{aligned}$$

as needed. ☐

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16 (Spring 19' Problem 6)

Determine the following integral:

$$\lim_{n\to\infty}\int_6^n (1+\frac{x}{n})^n e^{-2x}dx.$$

Proof. We will appeal to the DCT. First note that

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x.$$

Then we can define $f_n := (1+\frac{x}{n})^n e^{-2x}$ and it is pretty clear to see that

$$f_n \to \chi_{[6,\infty)}e^{-x}$$

a.e., then the integrable function g can be defined as $g(x) = \chi_{[0,\infty)}e^{-x}$ and we clearly have that $g(x)$ is an upper bound for the f_n thus by the DCT, the given integral converges to the integral of $\chi_{[6,\infty)}e^{-x}$, that is,

$$\lim_{n\to\infty}\int_6^n (1+\frac{x}{n})^n e^{-2x}dx = \int_6^\infty e^{-x}dx$$

and the RHS here is just e^{-6}. □

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17 (Fall 19' Problem 8)

If $f \in L^1(\mathbb{R}^d)$, then

$$||f_h - f|| \to 0,$$

as $h \to 0$.

Proof. Note first that simple, step, and continuous functions of compact support are all dense in $L^1(\mathbb{R}^d)$. In particular, for any $\epsilon > 0$ we can find a g continuous of compact support such that,

$$||f - g|| < \epsilon.$$

□

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18 (Spring 19' Problem 4)

Give definition of Leb measurable and show the open set defintion holds as well.

Proof. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. By definition we have that E^c is Lebesgue measurable as well. Then for any $\epsilon > 0$ we are guaranteed the existence of an open set $U \supset E^c$ such that

$$m(U \setminus E^c) < \epsilon.$$

As $U \subset \mathbb{R}^d$ is open, by definition we have that U^c is closed. Moreover, since $E^c \subset U$ we know $U^c \subset E$ and so we have

$$\begin{aligned} m(E \setminus U^c) &= m(U \setminus E^c) \\ &< \epsilon, \end{aligned}$$

as needed. Additionally we can show that

$$m(E) = \sup_{F \subset E, \text{closed}} \{m(F)\}$$

□

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19 (Fall 19' Problem 9)

Prove the BCT using Egorov.

Proof.

Egorov's Theorem: Let $\{f_n\}$ be a sequence of measurable functions define on a finite measure set E. If $f_n \to f$, then for any $\epsilon > 0$ there exists a closed set $A_\epsilon \subset E$ such that $f_n \to f$ uniformly on A_ϵ and $m(E \setminus A_\epsilon) < \epsilon$.

BCT: Let $\{f_n\}$ be a sequence of bounded measurable functions supported on a finite measure set E. If $f_n \to f$, then f is measurable, bounded, supported on E and

$$\int |f_n - f| \to 0.$$

So let us suppose Egorov's holds. Let $\{f_n\}$ be a sequence of measurable functions supported on a finite measure set. Note that $f(x)$ is bounded by M and vanishes for every $x \in E^c$. Since $f_n \to f$, for any $\epsilon > 0$ we are guaranteed the existence of a closed set $A_\epsilon \subset E$ such that

$$f_n \to f$$

uniformly on A_ϵ and that

$$m(E \setminus A_\epsilon) \leq \epsilon.$$

Thus for large n, we have for every $x \in A_\epsilon$ that

$$|f_n - f| \leq \epsilon.$$

Then we have

$$\begin{aligned} \int |f_n - f| &\leq \int_{A_\epsilon} |f_n - f| + \int_{E \setminus A_\epsilon} |f_n - f| \\ &\leq \epsilon m(E) + 2M m(E \setminus A_\epsilon) \end{aligned}$$

and ϵ was arbitrary thus we are done. □

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20. Functions tending to zero with non zero integral on $[0,1]$ (UCR QUAL)

Prove or disprove: If $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions such that for every $x \in [0,1]$,

$$\lim_{n\to\infty} f_n(x) = 0,$$

then

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

Proof. Disprove. Consider the sequence of functions

$$f_n(x) := \frac{1}{n}\sin(n^2 x).$$

diverges for $x \in [0,1]$. □

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21.Difference of f_h and f tends to zero (UCR QUAL)

Let f be integrable on $\mathbb{R}$. Prove that

$$\lim_{h \to 0} |f(x+h) - f(x)| dx = 0.$$

Proof. □

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22. Absolute continuity of Lebesgue integral

If f is Lebesgue integrable on $\mathbb{R}^d$, then for every given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_E |f| < \epsilon$$

whenever $m(E) < \delta$

Proof. We can safely suppose that $f \geq 0$. Then define

$$f_N(x) := f(x) \cdot \chi_{E_N},$$

where

$$E_N := \{x \in E : f(x) \leq N\}.$$

Then clearly the f_N are measurable. Furthermore, one has for every $N \in \mathbb{N}$ that

$$f_N \leq f_{N+1}.$$

Then by the Monotone convergence theorem, for any given $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\int f - f_N < \frac{\epsilon}{2}; \qquad (*).$$

Next choose δ such that $\delta < \frac{\epsilon}{2N}$. Thus if $m(E) < \delta$, we can compute

$$\begin{aligned}
\int_E f &= \int_E f - f_N + \int_E f_N && \text{adding 1} \\
&\leq \int f - f_N + \int_E f_N && \text{monotonicity of Lebesgue integral} \\
&\leq \int f - f_N + m(E)N && \text{by definition} \\
&\leq \int f - f_N + \frac{\epsilon}{2} && \text{as } m(E) < \delta \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{by } (*) \\
&= \epsilon && \text{adding fractions}
\end{aligned}$$

and as ϵ was arbitrarily chosen, we are done. □

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23. Bound on Hardy Littlewood maximal function

Let

$$f^*(x) := \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|dy.$$

Prove then that

$$m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha}||f||_{L^1}.$$

Proof. First a small lemma:

> **Vitalli's Covering Lemma** Suppose $B_1, ..., B_N$ is a finite collection of open balls. Then there exist a finite disjoint subcollection $B_{i_1}, ..., B_{i_n}$ such that
>
> $$m\left(\bigcup_{k=1}^{N} B_k\right) \leq 3^d \sum_{j=1}^{n} m(B_{x_j}).$$

Back to our proof. Define

$$E_\alpha := \{x \in \mathbb{R}^d : f^*(x) > \alpha\}.$$

Then for each $x \in E_\alpha$, one can find an open ball B_x containing x such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|dy > \alpha,$$

then for each x one has

$$m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)|dy; \qquad (*).$$

Next, as Lebesgue measure is a regular measure, one can show the inequality holds for an arbitrary compact subset of E_α. Thus we can fix some compact subset $K \subset E_\alpha$. Then clearly $\{B_x\}_{x\in E_\alpha}$ is a cover for K which is compact thus there exists some finite subset $A \subset E_\alpha$ such that

$$K \subset \bigcup_{x\in A} B_x.$$

Hence we have a finite collection of open balls, namely $\bigcup_{x\in A} B_x$. Then by our Covering Lemma, there exists a finite disjoint sub-collection, call it $B_{x_1}, ..., B_{x_N}$ such that

$$m\left(\bigcup_{x\in A} B_x\right) \leq 3^d \sum_{j=1}^{N} m(B_{x_j}); \qquad (**).$$

Then one can easily compute out

$$\begin{aligned}
m(K) &\leq m\bigg(\bigcup_{x\in A} B_x\bigg) && \text{Monotonicity of measurable sets} \\
&\leq 3^d \sum_{j=1}^{N} m(B_{x_j}) && \text{by } (**) \\
&\leq \frac{3^d}{\alpha} \sum_{j=1}^{N} \int_{B_{x_j}} |f(y)|dy && \text{by}(*) \\
&= \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{N} B_{x_j}} |f(y)|dy && \text{by disjointness of the } B_{x_j} \\
&\leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)|dy && \text{Monotonicity of Lebesgue integral}
\end{aligned}$$

and since this holds for some fixed compact subset under a regular measure, this holds in general and we are done. □

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24. Integral of composition equals integral of greater than set times derivative

Let $f : [0,1] \to [0,1]$ be continuous and $\phi : \mathbb{R} \to \mathbb{R}$ be C^1 with $\phi(0) = 0$. Prove that

$$\int_{[0,1]} \phi \circ f dx = \int m(\{x \in [0,1] : f(x) > t\})\phi'(t)dt.$$

Proof. We compute the RHS:

$$\begin{aligned}
\int m(\{x \in [0,1] : f(x) > t\})\phi'(t)dt &= \int \chi_{\{x \in [0,1] : f(x) > t\}} dx \phi'(t)dt && \text{definition of Lebesgue integral} \\
&= \int \chi_{\{t : f(x) > t\}} dx \phi'(t)dt && \text{unsure} \\
&= \int \chi_{\{t : f(x) > t\}} \phi'(t)dtdx && \text{Fubini's Theorem} \\
&= \int \phi(f(x))dx && \text{integrating } \phi'(t) \text{ from 0 to } f(x)
\end{aligned}$$

as needed. □

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25. Convergence in L^2 implies convergent subsequence (UCR QUAL)

Let $\{f_n\} \subset L^2(\mathbb{R})$ converge to 0. Prove f_n has a subsequence that converges to 0 a.e.

Proof. First a small lemma:

Convergence in measure implies convergence a.e. of a subsequence.

We show that f_n converges to 0 in measure. That is, show

$$m(\{x : |f_n(x)| > \epsilon\}) \to 0$$

as n tends to ∞. Convergence in L^2 implies

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n|^2 dm = 0.$$

Set $A_\epsilon(n) := \{x \in \mathbb{R} : f_n(x) > \epsilon\}$, then one can compute

$$\begin{aligned}
\int_{\mathbb{R}} |f_n|^2 dm &> \int_{A_\epsilon(n)} |f_n|^2 dm && \text{monotonicity of Lebesgue integral} \\
&> \int_{A_\epsilon(n)} \epsilon^2 dm && \text{by assumption} \\
&= \epsilon^2 m(A_\epsilon(n)) && \text{definition of Lebesgue integral} \\
&= \epsilon^2 m(\{x : |f_n(x)| > \epsilon\}) && \text{by definition}
\end{aligned}$$

Thus dividing by ϵ^2 we obtain

$$m(\{x : |f_n(x)| > \epsilon\}) < \frac{||f_n||_2}{\epsilon^2}$$

as needed. Then by our Lemma, as the f_n converge in measure, they have a subsequence converging to 0 a.e. □

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26. Convergence in measure implies convergent subsequence a.e. (UCR QUAL)

Show if $\{f_n\}$ is a sequence converging to f, then there exists subsequence of f_n converging to f a.e.

Proof. Suppose $f_n \xrightarrow{\text{measure}} f$. That is,

$$m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0; \qquad \text{as } n \to \infty.$$

Given any $\epsilon > 0$, setting $\epsilon = 2^{-k}$ for $k \in \mathbb{N}$ we can choose n_k such that

$$m(\{x : |f_n(x) - f(x)| > 2^{-k}\}) \leq 2^{-k}, \qquad (*)$$

whenever $n \geq n_k$. Without any loss of generality, suppose $n_{k+1} \geq n_k$ for all $k \in \mathbb{N}$. Define

$$A := \{x : |f_{n_k}(x) - f(x)| > 2^{-k}\}.$$

By $(*)$ we have that

$$\sum_{k=1}^{m}(A_k) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty,$$

Then since the sum of the measures is $m-$finite, by the Borell-Cantelli lemma, we have

$$m\left(\lim_{r\to\infty} \sup A_k\right) = 0,$$

this forces

$$\lim_{k\to\infty} f_{n_k}(x) = f(x),$$

as desired. □

27. For $f \in L^1(0,\infty)$ there exists a sequence x_n such that $x_n f(x_n) \to \infty$ as $n \to \infty$.

Show if $f \in L^1(0,\infty)$, there exist a sequence $x_n \to \infty$ such that

$$\lim_{n\to\infty} x_n f(x_n) = 0.$$

Proof. Let

$$c := \lim \inf_{x\to\infty} x|f(x)|.$$

If $c = 0$, then a sequence exists. If $c > 0$, then there exists some $A > 0$ such that

$$x|f(x)| \geq \frac{c}{2}; \qquad \text{for all} x > A.$$

Then we have

$$\begin{aligned} \int_{(0,\infty)} |f| dm &\geq \int_{(A,\infty)} |f| dm \\ &\geq \int_A^\infty \frac{c}{2x} dx \\ &= \frac{c}{2} \ln(x) \Big|_A^\infty \\ &= \infty. \end{aligned}$$

This contradicts $f \in L^1(0,\infty)$ which implies $\int_0^\infty |f| dm < \infty$. □

28.

Show if $f \in L^1((0,1)$ and g is defined via

$$g(x) = \int_x^1 \frac{f(t)}{t} dt$$

Prove $g \in L^1((0,1))$.

Proof. Differentiating we obtain

$$g'(x) = \frac{f(x)}{x}$$

this implies

$$f(x) = -xg'(x)$$

Then integrating we obtain

$$\int_0^1 f(x)dx = \int_0^1 -xg'(x)dx$$

Integrating by parts using

$$u = -x, du = -1$$
$$v = g(x), dv = g'(x)$$

We have that

$$\begin{aligned} \int_0^1 -xg(x)dx &= -xg(x)\Big|_0^1 + \int_0^1 g(x)dx && \text{integration by parts} \\ &= \int_0^1 g(x)dx && \text{since } g(1) = 0 \end{aligned}$$

Thus

$$\int_0^1 g(x) = \int_0^1 f(x)dx < \infty$$

as needed forcing $g \in L^1((0,1))$. □

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