Informal Notes: Several Complex Variables

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Complex Geometry by Daniel Huybrechts;
Functions of One Complex Variable I by John B. Conway;
Algebraic Curves and Riemann surfaces by Rick Miranda
Principles of Algebraic Geometry by Phillip Griffiths and Joseph Harris

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Abstract

In these notes we give a brief overview at the theory of functions of several complex variables. The only prior knowledge to have to make progress in these notes is basic point set topology, introductory Complex Analysis (analytic functions, Cauchy-Riemann equations, Liouvilles Theorem.), Abstract Algebra, and some PDEs. Our goal is to approach the theory first from that of a single complex variable then generalize to higher dimensions whilst trying to not loose track of any complex or algebraic structure.

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§1. Holomorphic Functions

The theory of functions of several complex variables is a rich and elegant field; it pulls from various fields such as but not limited to, Topology, Abstract Algebra, and Partial differential equations. One of the most crucial concepts across the board for complex variables is the notion of a *holomorphic* map. Before moving into the land of several complex variables, we must first recall what it means for a function defined over $\mathbb{C}$ to be holomorphic. For the remainder of these notes, $U \in \tau_{\mathbb{C}}$ when defined. That is, U is open in $\mathbb{C}$. Also we will consider the reader knows that $\mathbb{C}$ is a field and under the usual product operations we have that $i = \sqrt{-1}$ which can be left as an exercise for the reader. Additionally, note for $z \in \mathbb{C}$ one can write

$$z = x + iy,$$

where $x, y \in \mathbb{R}$. Then one can define the *complex conjugate* of z by

$$\overline{z} = x - iy.$$

Exercise 0. Show in $\mathbb{C}$, $i = \sqrt{-1}$. (Hint: $(a, b)(c, d) = (ac - bd, ad + bc)$).

Let $U \subset \mathbb{C}$. We say that a map

$$f : U \to \mathbb{C}$$

is *holomorphic* if for each $z \in U$,

$$\frac{\partial f}{\partial \overline{z}} = 0$$

Equivalently, one could say the limit

$$\lim_{h \to \infty} \frac{f(z + h) - f(z)}{h}$$

exists. Note that here $0 \neq h \in \mathbb{C}$ (Here, h could tend to zero from an uncountable number of directions. This is one of the big reasons why being continuous over $\mathbb{C}$ is much stronger than being continuous over $\mathbb{R}$. Over the reals one only need to ensure the left and right sided limits exists and are identical, over the complex plane however, a point can be approached from any of the angles within 360°.) and $z + h \in U$.

We say f is *analytic* on U if for each $z_0 \in U$ there exists an $\epsilon > 0$ such that $f(z)$ converges on $B_\epsilon(z_0)$. Moreover, f is analytic if it can be written as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$$

for every $z \in B_\epsilon(z_0)$.

Theorem 0. *Let $f \in C^\infty(U)$. Then f is holomorphic if and only if f is analytic.*

Proof. First let us suppose f is holomorphic on U. That is, for every $z \in U$ we have that

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This by definition is a limit thus we can find and $\epsilon > 0$ such that for $z_0 \in U$ and $z \in B_\epsilon(z_0)$, by the Cauchy integral formula we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(w)}{w-z}dw \\ &= \int_{\partial B_\epsilon(z_0)} \frac{f(w)}{(w-z_0)-(z-z_0)}dw \\ &= \int_{\partial B_\epsilon(z_0)} \frac{f(w)}{(w-z_0)-(1-\frac{z-z_0}{w-w_0})}dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n. \end{aligned}$$

To see why we can apply Cauchy's Integral formula, we typically integrate along a path or simple closed curve in the complex plane. In our case, our curve

$$\gamma : [a,b] \to \mathbb{C}$$

is just the boundary of our ϵ ball around z_0. Thus setting the coefficients equal to

$$a_n = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw,$$

we have that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for $z \in B_\epsilon(z_0)$ where the sum converges uniformly and absolutely on any small disk.

On the other hand, suppose f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

which converge for every $z \in U$ which in our case we can let be $B_\epsilon(z_0)$. Since the partial with respect to the conjugate

$$\frac{\partial}{\partial \overline{z}}(z-z_0)^n = 0$$

the partial sums satisfy Cauchy's Integral formula, without the area integral, and by uniform convergence of the sum in a ball centered at z_0, the same is true of f, i.e.,

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(w)}{z-w}dw$$

we can then differentiate inside the integral as follows

$$\frac{\partial}{\partial \overline{z}}f(z) = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{\partial}{\partial \overline{z}}\left(\frac{f(w)}{z-w}\right)dw = 0$$

Since for $z \neq w$

$$\frac{\partial}{\partial \overline{z}}\left(\frac{1}{w-z}\right) = 0$$

□

There are many equivalent definitions of holomorphicity (or in our case now analyticity) over $\mathbb{C}$, however one definition we appeal to most is that of the *Cauchy-Riemann* equations. Recall that $z \in \mathbb{C}$ can be written as

$$z = x + iy$$

where $x, y \in \mathbb{R}$. Then f can be regarded as a function $f(x, y)$ of two real variables. As a matter of fact one can write f as

$$f(x, y) = u(x, y) + iv(x, y)$$

where

$$u(x, y) := \mathrm{Re}(f), v(x, y) := \mathrm{Im}(f),$$

the real and imaginary parts respectively. Note that u, v are real-valued functions

$$u, v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Now one can show that f is holomorphic if and only if the Cauchy-Riemann equations are satisfied. That is, if and only if

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

I.e., the derivative of f need be $\mathbb{C}-$linear. This allows us to define differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \qquad (*)$$

and

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}). \qquad (**)$$

These are motivated by the properties

$$\frac{\partial}{\partial z}(z) = 1 = \frac{\partial}{\partial \overline{z}}(\overline{z}),$$

and

$$\frac{\partial}{\partial z}(\overline{z}) = 0 = \frac{\partial}{\partial \overline{z}}(z),$$

Then the C-R equations can be writtten as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

As the jump from real partial derivatives to complex partial derivatives is vast, we will spend a bit more time on this section. Consider a differentiable map

$$f : U \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2.$$

Then it only makes sense to mention the differential of f at some $z \in U$. Namely, its differential $df(z)$ at some $z \in U$ is the $\mathbb{R}-$linear map

$$df(z) : T_z\mathbb{R}^2 \to T_{f(z)}\mathbb{R}^2$$

between tangent spaces. It is crucial to note the dimension of the tangent space is the same as the dimension of our ambient space. Writing $z = x + iy, w = r + is$ for the two tangent spaces we enjoy a nice canonical bases given via

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle, \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial s} \rangle.$$

With respect to these basis, the differential $df(z)$ is given via the real Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

where $f = u + iv$. I.e., $u = r \circ f$, $v = s \circ f$.

Our goal is to now extend this to a $\mathbb{C}-$linear map, (Recall a map f is $\mathbb{C}-$ linear if

$$f(i) = if(1).)$$

which is given via

$$df(z)_{\mathbb{C}} : T_z\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z)}\mathbb{R}^2 \otimes \mathbb{C}.$$

We can now choose as basis, $(*)$ and $(**)$. With respect to this basis, $df(z)$ is given via

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \\ \frac{\partial \overline{f}}{\partial z} & \frac{\partial \overline{f}}{\partial \overline{z}} \end{pmatrix}.$$

Exercise 1. Show that for any function f defined one some open $U \subset \mathbb{C}$,

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\left(\frac{\partial f}{\partial z}\right)}.$$

Exercise 2. If $f = u + iv$ is holomorphic with

$$u, v : \mathbb{R}^2 \to \mathbb{R},$$

and

$$u, v \in C^\infty$$

then

$$\frac{\partial f}{\partial \overline{z}} = 0 = \frac{\partial \overline{f}}{\partial z}.$$

Using results from these two exercises we note that $df(z)$ has a new base given via the diagonal matrix

$$\begin{pmatrix} \frac{\partial f}{\partial z} & 0 \\ 0 & \frac{\partial \overline{f}}{\partial \overline{z}} \end{pmatrix}.$$

Holomorphicity of f is also equivalent to the *Cauchy Integral Formula*. More precisely, a function

$$f : U \to \mathbb{C}$$

is holomorphic if and only if f is continuously differentiable and for any

$$B_\epsilon(z_0) \subset U,$$

we have that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{(z - z_0)} dz.$$

In fact, this formula holds for any function

$$f : \overline{B_\epsilon(z_0)} \to \mathbb{C}$$

which is holomorphic on the interior. Namely, the Cauchy integral formula is used in proving the existence of a (convergent) power series expansion of any function satisfying the Cauchy-Riemann equations.

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We list a few crucial results from functions of a single variable which will be of use to us. The first three apply to to $\mathbb{C}^n$ for every $n \in \mathbb{N}$. However, the mapping theorem and extension theorem only work for $n = 1$ and $n \geq 2$ respectively. This is one of the biggest issues in taking a leap from $\mathbb{C}$ to $\mathbb{C}^n$.

Maximum Principle *Let $U \subset \mathbb{C}$ be open and connected. If*

$$f : U \to \mathbb{C}$$

is holomorphic and non-constant, $|f|$ has no local maximum in U. Moreover, if U is bounded and f can be extended to a continuous function

$$\overline{f} : \overline{U} \to \mathbb{C},$$

then $|f|$ takes on its maximal values on ∂U.

Maximum Principle (Alternate) *Let $U \subset \mathbb{C}$ is open and connected If*

$$f : U \to \mathbb{C}$$

is holomorphic and there exists a point $z_0 \in U$ such that

$$|f(z_0)| \geq |f(z)|$$

for every $z \in U$, then f is constant on U.

Identity Theorem *If*

$$f, g : U \to \mathbb{C}$$

are holomorphic functions on an open and connected subset $U \subset \mathbb{C}$ *such that*

$$f(z) = g(z)$$

for each $z \in V \subset U$ *some non-empty open subset, then* $f = g$.

Riemann extension Theorem *Let*

$$f : B_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$$

bee a bounded holomorphic function. Then, f *can be extended to a holomorphic function*

$$f : B_\epsilon(0) \to \mathbb{C}.$$

(*Note this only holds for the case* $n \geq 2$.)

Riemann Mapping Theorem *Let* $U \subsetneq \mathbb{C}$ *be simply connected. Then* $U \cong B_1(0)$. *That is, there exists a bijective holomorphic map*

$$f : U \to B_1(0)$$

such that f^{-1} *is also holomorphic.*

In other words, U is conformally equivalent to the open unit disk if U is not all of $\mathbb{C}$ (Tao, 2018). (*Note this only holds for the case* $n = 1$. *To see why it fails for higher dimensions refer to this exercise*)

Liouville *If*

$$f : \mathbb{C} \to \mathbb{C}$$

is bounded, then f *is constant. I.e., Bounded entire functions need be constant. Here entire is used to denote holomorphicity on all of* $\mathbb{C}$.

Note if we swap $\mathbb{C}$ *for* $\mathbb{R}$ *this does not hold. To see this, consider*

$$f : \mathbb{R} \to \mathbb{R}$$

defined via

$$x \mapsto \sin x$$

which is clearly bounded, real-analytic but not constant (over $\mathbb{R}$.

Residue Theorem *Let*

$$f : B_\epsilon(0) \setminus \{0\} \to \mathbb{C}$$

be a holomorphic map. Then f *can be extended via a Laurent Series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where the coefficient a_{-1} *is given by the residue formula*

$$a_{-1} = \frac{1}{2\pi i} \int_{|z| = \frac{\epsilon}{2}} f(z) dz.$$

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We can now safely extend our notions to functions of several complex variables. The notion can be extended in two ways. Firstly, one could consider a function of several complex variables

$$\mathbb{C}^n \to \mathbb{C}.$$

Secondly, functions taking on values actually in $\mathbb{C}^n$. We must first consider what the correct basis choice would be for higher (complex) dimensions. As a basis for the topology in higher dimensions we will use these *polydisks*

$$B_\epsilon(w) = \{z : |z_j - w_j| < \epsilon_j\},$$

where $\epsilon := \{\epsilon_1, ..., \epsilon_n\}$.

Definition 1.1 Let $U \subset \mathbb{C}$. Let

$$f : U \to \mathbb{C}$$

be continuously differentiable. Then f is said to be *holomorphic* if the Cauchy-Riemann equations hold for all coordinates

$$z_j = x_j + iy_j.$$

I.e.,

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}$$

and

$$\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$$

for $j \in \{1, 2, ..., n\}$.

So by definition a continuously differentiable function f is holomorphic if the induced functions

$$f|_U : U \cap \{(z_1, z_2, ..., z_{j-1}, z, z_{j+1}, ..., z_n) : z \in \mathbb{C}\} \to \mathbb{C}$$

are holomorpic for every choice of j and for fixed $z_1, z_2, ..., z_{j-1}, z, z_{j+1}, ..., z_n \in \mathbb{C}$.

More simply put, a functions holomorphicity depends heavily on its component-wise holomorphicity as j ranges from 1 to n about each point $z \in \mathbb{C}$.

We can now rewrite our differential operators component-wise as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

and

$$\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

By the Cauchy-Riemann equations we have that

$$\frac{\partial f}{\partial \overline{z_j}} = 0$$

for $j \in \{1, 2, ..., n\}$. One tricky observation is that the equations in Definition 1.1 yield

$$\overline{\partial} f = 0.$$

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We next turn our attention to the Cauchy integral formula for functions of several variables. Let us first Recall what this means in $\mathbb{C}$:

Cauchy's Integral Formula (for $\mathbb{C}$) *Let $U \subset \mathbb{C}$ and*

$$f : U \to \mathbb{C}$$

holomorphic. Let γ is a closed curve in U. That is,

$$\gamma : [a, b] \to \mathbb{C}$$

is a continuous map with

$$\gamma(a) = \gamma(b).$$

Suppose $\eta(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus U$, then for $z_0 \in U \setminus \gamma([a, b])$ we have

$$\int_\gamma \frac{f(z)}{z - z_0} dz = (2\pi i)\eta(\gamma; z_0)f(z_0)$$

where

$$\eta(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz \in \mathbb{Z},$$

is the winding number of γ about z_0. Thus if the winding number about a particular point is 0, the value of the integral is therefore 0 as well.

Cauchys Integral Formula for ($\mathbb{C}^n$) *Let*

$$f\overline{B_\epsilon(w)} \to \mathbb{C}$$

be a continuous function such that f is holomorphic with respect to each component z_j in any point $z \in B_\epsilon(w)$. Then for any $z \in B_\epsilon(w)$,

$$\int_{|\xi_j - w_j| = \epsilon_j} \frac{f(\xi_1, \xi_2, ..., \xi_n)}{(\xi_1 - z_1)(\xi_2 - z_2)...(\xi_n - z_n)} d\xi_1 d\xi_2 \cdots d\xi_n = (2\pi i)^n f(z).$$

Proof. Repeated application of the Cauchy integral formula for the one variable case allows us to swap the iterated integral for the multiple integral. □

Thus continuous functions on open domains which are holomorphic with respect to each single coordinate (whilst others remain fixed) need be holomorphic! This is more famously known as Osgood's Lemma. The lemma is the specific case of Hartog's Theorem which we will get to momentarily. This result drops the assumption that the function need be continuous). As in the case for $\mathbb{C}$, the above integral can be used in writing out a power series expansion of any holomorphic function $f : U \to \mathbb{C}$. More precisely, for any $w \in U$, there exists a polydisk

$$B_\epsilon(w) \subset U \subset \mathbb{C}^n$$

such that the restriction $f|_{B_\epsilon(w)}$ is given via the power series

$$\sum_{j_1,\dots,j_n=0}^{\infty} a_{j_1,j_2,\dots,j_n}(z_1-w_1)^{j_1}\cdots(z_n-w_n)^{j_n}$$

with coefficients given via

$$a_{j_1,\dots,j_n} = \frac{1}{j_1!\cdots j_n!}\cdot\frac{\partial^{j_1+\dots+j_n}f}{\partial z_1^{j_1}\cdots\partial z_n^{j_n}}.$$

From our above list of fun facts and theorems, the maximum principle, identity theorem, and Liouville generalize rather nicely into higher dimensions. A particular version of the Riemann Extension Theorem holds true and requires proof. The Riemann mapping theorem fails in higher dimensions however. To see the later, consider the the unit disk in $\mathbb{C}^2$ and the polydisk

$$B_{(1,1)}(0) \subset \mathbb{C}^2$$

which are not biholomorphic to one another. The proof is left as an exercise.

Exercise 3. Show the polydiskand the unit disk are not equivalent. That is, show there does not exits a biholomorphic map between

$$B_n := \{z \in \mathbb{C}^n : ||z|| < 1\}$$

and

$$D_n := \{z \in \mathbb{C}^n : |z_j| < 1\},$$

where $j = 1, 2, ..., n$.

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Often times the holomorphicity of a function of several variables is shown by representing a function as an integral of a known holomorphic function. The following Lemma will be of use to use further down the line.

Lemma 1.2 *Let $U \subset \mathbb{C}^n$. Also let $V \subset \mathbb{C}$ be an open neighborhood of the boundary of $B_\epsilon(0) \subset \mathbb{C}$. Assume that*

$$f : V \times U \to \mathbb{C}$$

is a holomorphic function. Then

$$g(z) := g(z_1, ..., z_n) := \int_{|\zeta|=\epsilon} f(\xi, z_1, ..., z_n)d\xi$$

is holomorphic on U.

Proof. Let $z \in U$. If $|\xi| = \epsilon$ then there exists a polydisk

$$B_{\delta(\xi)}(\xi) \times B_{\delta'(\xi)}(z) \subset V \times U$$

on which f has a power series expansion. As $\partial B_\epsilon(0)$ is compact we can find a finite number of points, call them

$$\xi_1, ..., \xi_n \in \partial B_\epsilon(0)$$

and positive real numbers

$$\delta(\xi_1), ..., \delta(\xi_n)$$

such that

$$\bigcup (\partial B_\epsilon(0) \cap B_{\frac{\delta(\xi_j)}{2}}(\xi_j))$$

is a disjoint union and

$$\partial B_\epsilon(0) = \bigcup (\partial B_\epsilon(0) \cap \overline{B_{\frac{\delta(\xi_j)}{2}}(\xi_j)}).$$

Hence,

$$\begin{aligned} g(z) &= \int_{|\xi|=\epsilon} f(\xi, z_1, ..., z_n) d\xi \\ &= \sum_{j=1}^{k} \int_{|\xi|=\epsilon, |\xi_j - \xi| < \frac{\delta(\xi_j)}{2}} f d\xi. \end{aligned}$$

Thus each summand is holomorphic since the power series expansion of f converges uniformly on

$$\overline{B_{\frac{\delta(\xi_j)}{2}}(\xi_j)}$$

and thus can be swapped with the integral. □

The next result is a key result in the existence of a holomorphic extension without the assumption the function in question is continuous. Also note this result is only valid in $\mathbb{C}^n$ for $n \geq 2$.

Proposition 1.3: Hartog's Theorem *Let* $\epsilon = (\epsilon_1, ..., \epsilon_n)$, $\epsilon' = (\epsilon'_1, ..., \epsilon'_n)$ *be given such that for each* $1 \leq j \leq n$ *we have*

$$\epsilon'_j < \epsilon_j.$$

If $n > 1$, *then any holomorphic map*

$$f : B_\epsilon(0) \setminus \overline{B_{\epsilon'(0)}} \to \mathbb{C}$$

can uniquely be extended to a holomorphic map

$$f : B_\epsilon(0) \to \mathbb{C}.$$

Proof. It suffices to assume $\epsilon = (1, ..., 1)$. That is, $B_\epsilon(0) \subset \mathbb{C}^n$ is the unit polydisk. As $\epsilon'_j < \epsilon_j$ for each j, there exists $\delta > 0$ such that the open set

$$V := \{z \in \mathbb{C}^n : 1 - \delta < |z_1| < 1, |z_{j \neq 1}| < 1\} \cup \{z \in \mathbb{C}^n : 1 - \delta < |z_2|, |z_{j \neq 2}| < 1\}$$

is properly contained in the complement

$$B_\epsilon(0) \setminus B_{\epsilon'}(0).$$

Note that this gives us an annulus "in between" z_1, z_2. In particular,

$$f : V \to \mathbb{C}$$

is holomorphic. Thus, for any

$$w := (z_2, ..., z_n) \in \mathbb{C}^{n-1},$$

with $|z_j| < 1$, we are guaranteed the existence of a holomorphic function

$$f_w(z_1) := f(z_1, z_2, ..., z_n) \in \mathbb{C}^n$$

on

$$\{z \in \mathbb{C}^n : 1 - \delta < |z_1| < 1, |z_{j \neq 1}| < 1\}.$$

Let

$$f_w(z_1) := \sum_{n=-\infty}^{\infty} a_n(w) z_1^n$$

be the Laurent series of this function. Then the coefficients are given by

$$a_n(w) = \frac{1}{2\pi i} \int_{|\xi|=1-\frac{\delta}{2}} \frac{f_w(\xi)}{\xi^{n+1}} d\xi$$

(this is a result of the Laurent series development) which is holomorphic for w in the unit polydisk of $\mathbb{C}^{n-1}$ by our Lemma. So for a fixed $w \in \mathbb{C}^{n-1}$, the map given via

$$z_1 \mapsto f_w(z_1)$$

is holomorphic on the unit polydisk such that

$$1 - \delta < |z_2| < 1.$$

Thus the negative coefficients are all zero. This forces the coefficients to be identically zero. We can now extend f holomorphically to $\overline{f}$ by the following power series:

$$\sum_{n=0}^{\infty} a_n(w) z_1^n.$$

As the $a_n(w)$ are holomorphic, they attain their max on the boundary. Thus convergence on our annulus yields uniform convergence everywhere. The holomorphic function given in our series glues together with f to give us the desired holomorphism. □

It should be noted that our previous result fails in the one variable case. Informally, Hartog's Theorem says the singularities of f are given by vanishing sets of holomorphic functions.

Next we will examine the Weierstrass preparation theorem (WPT), which will play a key role in the theory of functions of several complex variables.

Let

$$f : B_\epsilon(0) \to \mathbb{C}$$

be holomorphic on the polydisk $B_\epsilon(0)$. For any $w = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$ we let $f_w(z_1)$ denote the function $f(z_1, ..., z_n)$ over $\mathbb{C}^n$. We show that the zeroes of f are caused by a factor of f which has the form of a Weierstrass polynomial.

Definition 1.5 A *Weierstrass polynomial* in z_1 of the form

$$\sum_{j=0}^{d} \alpha_j(w) z_1^{d-j}$$

where the $a_j(w)$ are each holomorphic functions on some small polydisk in $\mathbb{C}^{n-1}$ such that they vanish at the origin.

Recall that any holomorphic function of one complex variable with a zero of order d can be written as

$$z^d h(z),$$

with $h(0) \neq 0$.

Note a zero of order d of $f_0(z_1)$ could decompose into a collection of zeroes of $f_w(z_1)$ whose orders ass up to d. The following generalizes the notion of local normal form for the one variable case.

Proposition 1.4: Weierstrass Prep Theorem Let

$$f : B_\epsilon(0) \to \mathbb{C}$$

be holomorphic on the polydisk $B_\epsilon(0)$. Assume $f(0) = 0$ and $f_0(z_1) \neq 0$. Then there exists a (unique) Weierstrass Polynomial

$$g(z_1, w) = g_w(z_1)$$

and a holomorphic function h on some smaller polydisk $B_{\epsilon'}(0) \subset B_\epsilon(0)$ such that

$$f = g \cdot h$$

and $h(0) \neq 0$.

Proof. Let f be given. By the fundamental theorem, we know f has zeroes over the complex number. Thus we can let the multiplicity of f be d with zeroes

$$a_1(w), ..., a_d(w).$$

Since f_0 is not identically zero, we can find an $\epsilon_1 > 0$ such that $f_0 \in \overline{B_{\epsilon_1}(0)}$ vanishes only in 0. Then choose $\epsilon_2, ..., \epsilon_n > 0$ such that

$$f(z_1, z_2, ..., z_n) \neq 0$$

for $|z_1| = \epsilon_1$ and $|z_j| < \epsilon_j$ for $j = 2, ..., n$. Note if $w = 0$ then

$$a_1(0) = ... = a_d(0) = 0.$$

Here each zero occurs as much as its multiplicity determines. Next, it would be nice if we knew if there was a relationship between d and w. That is, if d depends on w or not. To see this, consider following polynomial (Which has the same zeroes, with multiplicities as $f_w(z_1)$),

$$g_w(z_1) := \prod_{j=1}^{d} (z_1 - a_j(w)).$$

Thus, for a fixed w, the function

$$h_w(z_1) := \frac{f_w(z_1)}{g_w(z_1)}$$

is holomorphic in z_1. (As quotients are holomorphic provided $g'(z_1) \neq 0$). We must now only show $g_w(z_1), h_w(z_1)$ are holomorphic in w. To see this, note that the coefficients of $g_w(z_1)$ can be written as

$$\sum_{j=1}^{d} a_j(w)^k$$

for $k = 1, 2, ..., n$. Thus

$$g_w(z_1) - w(z_1) = \prod_{j=1}^{d}(z_1 - a_j(w)) - w(z_1)$$

is holomorphic in w given the above summands are holomorphic in w. We must apply the Residue Theorem to

$$z_1^k \frac{f'_w(z_1)}{f_w(z_1)}.$$

Let

$$f_w(\xi) = \sum_{j=m}^{\infty} a_j(\xi - a)^j$$

be the power series of f_w in some zero, a. Then it is clear to see that the derivative is given via

$$f'_w(\xi) = \sum_{j=m}^{\infty} j a_j(\xi - a)^{j-1}.$$

Moreover,

$$\xi^k = a^k + ka^{k-1}(\xi - a) + ...$$

Then from a first the residue theorem we get

$$\text{Res}_{\xi=a}(\xi^k \frac{f'_w(\xi)}{f_w(\xi)}) = ma^k.$$

Therefore, the polynomial expression og $g_w(z_1)$ can be written as

$$\sum_{j=1}^{d} a_j(w)^k = \frac{1}{2\pi i} \int_{|\xi|=\epsilon_1} \xi^k \frac{f'_w(\xi)}{f_w(\xi)} d\xi.$$

The left is holomorphic in w by our Lemma as f_w is holomorphic in w, thus $g_w(z_1)$ is holomorphic in $z_1, ..., z_n$. Note that when $k = 0$, the left side is the number of zeroes of f_w counted with multiplicities. Thus this integer d depends holomorphically on w and therefore does not depend on w at all.

Observe that

$$\{(z_1, w) : z_1 = a_j(w)\}^c$$

for some j, contains a neighborhood of

$$\{(z_1, w) : |z_1| = \epsilon_1, |z_{j\neq 1}| < \epsilon_j\}.$$

And so by the Cauchy integral formula

$$h_w(z_1) = \frac{1}{2\pi i}\int_{|\xi|=\epsilon_1} \frac{h_w(\xi)}{(\xi - z_1)} d\xi,$$

together with the holomorphicity of $\frac{f}{g}$ gives us that h is holomorphic everywhere, by our lemma.

For the uniqueness, since $h(0) \neq 0$, h does not vanish anywhere thus f_w, g_w have the same vanishing sets and the only Weierstrass polynomial with this property is polynomial we have just constructed. □

From here on, as a short hand we will let $Z(f)$ denote the *zero set* or vanishing set of a holomorphic function f. That is,

$$Z(f) = \{z : f(z) = 0\}.$$

Proposition 1.7 (Riemann extension theorem) Let f be holomorphic on some $U \subset \mathbb{C}^n$. If

$$g : U \setminus Z(f) \to \mathbb{C}$$

is holomorphic and locally bounded near $Z(f)$, then g can be uniquely extended to a holomorphic function

$$\overline{g} : U \to \mathbb{C}.$$

Proof. First we examine the special case for when $n = 2$ and $f(z) = z_1$. Then

$$g_{z_2}(z_1) := g(z_1, z_2)$$

is bounded and holomorphic on some punctured disk in the plane. That is, bounded and holomorphic near $Z(f)$. Thus we can find an extension of g_{z_2} to a holomorphic function on the whole disk. It would be left to show these functions all glue together in the piece-wise sense.

For $n \geq 3$, we may very well suppose that $U \subset \mathbb{C}$ is given by

$$U := B_\epsilon(0).$$

Let us define the line in $\mathbb{C}^n$ via

$$\mathcal{L}_1 := \{(z_1, 0, ..., 0) : z_1 \in \mathbb{C}\}$$

Note: $\mathcal{L}_j$ makes all coordinates zero other than z_j

Thus we may assume that

$$U \cap L \nsubseteq Z(f)$$

. Additionally we can even boil this down to the case that the restriction f_0 of f to $\mathcal{L}$ vanishes only at $(0, 0, ..., 0)$. Thus for when we set

$$|z_1| = \frac{\epsilon_1}{2}$$

we get that

$$f_0(z_1) \neq 0.$$

Then we could wiggle our $\epsilon_2, ..., \epsilon_n > 0$ in order to assume

$$f(z) \neq 0$$

given that

$$|z_1| = \frac{\epsilon_1}{2},$$

and

$$|z_{j\neq 1}| < \frac{\epsilon_j}{2}.$$

That is, for any given w with the above given condition above on the coordinates of w, the function f_w has no zeroes on

$$\partial B_{\frac{\epsilon_1}{2}}(0).$$

(Must recall what it means having no zeroes on the boundary By our assumption, the restriction g_w of g to $B_{\frac{\epsilon_1}{2}}(0) \setminus Z(f_w)$ is bounded and thus can be extended to a holomorphic function $\tilde{g}$ on $B_{\frac{\epsilon_1}{2}}(0)$. By Cauchy's infamous integral formula however, this extension is given via

$$\tilde{g}_w(z_1) = \frac{1}{2\pi i} \int_{\partial B_{\frac{\epsilon_1}{2}}(0)} \frac{g_w(\xi)}{\xi - z_1} d\xi.$$

As f_x has no zeroes on this boundary,

$$f_w(\xi) \neq 0$$

for any $\xi \in \partial B_{\frac{\epsilon_1}{2}}(0)$. In turn, the integrand is holomorphic in (z_1, w). By our Lemma 1.3 (once again),

$$\tilde{g}(z_1, w) := \tilde{g}_w(z_1)$$

is then holomorphic on (z_1, w) and we thus have the holomorphic of extension of g, namely this $\tilde{g}_w(z_1)$. □

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We now focus our attention to extend the notion of holomorphicity to function who take on values in $\mathbb{C}^n$. We do so as follows.

Definition 1.8 Let $U \subset \mathbb{C}^m$ be open.Then a map

$$f : U \to \mathbb{C}^n$$

is said to be *holomorphic* if each coordinate function $f_1, f_2, ..., f_n$ is a holomorphic function

$$f_j : U \to \mathbb{C}.$$

In an analogy to the one-dimensional case, we say

$$f : U \to V$$

where $U, V \subset \mathbb{C}^n$ is *biholomorphic* if and only if f is a bijective, holomorphic, with a holomorphic inverse.

Definition 1.8 Let $U \subset \mathbb{C}^m$ be open and let

$$f : U \to \mathbb{C}^n$$

be holomorphic. Then the (complex) Jacobian of f at some $z \in U$ is the matrix given via

$$J(f)(z) := \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{1\leq i\leq n, 1\leq j\leq m}.$$

Definition: A point $z \in U$ is called *regular* if $J(f)(z)$ is onto. If every point $z \in f^{-1}(w)$ is regular, then w is a *regular value*.

As in the one-dimensional case, it is useful to relate the complex Jacobian $J(f)$ to the real one. This goes as follows. The differentiable map

$$f : U \subset \mathbb{C}^m = \mathbb{R}^{2m} \to \mathbb{C}^n = \mathbb{R}^{2n}$$

induces for $z \in U$ the $\mathbb{R}-$linear map

$$df(z) : T_z\mathbb{R}^{2m} \to T_{f(z)}\mathbb{R}^{2n}.$$

With respect to the basis

$$\langle \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_m} \rangle$$

and

$$\langle \frac{\partial}{\partial r_1}, ..., \frac{\partial}{\partial r_m}, \frac{\partial}{\partial s_1}, ..., \frac{\partial}{\partial s_m} \rangle,$$

the linear map $df(z)$ is given by the real Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j} & \left(\frac{\partial u_i}{\partial y_j}\right)_{i,j} \\ \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j} & \left(\frac{\partial v_i}{\partial y_j}\right)_{i,j} \end{pmatrix}.$$

The $\mathbb{C}-$linear extension

$$d(f)_{\mathbb{C}} : T_z\mathbb{R}^{2m} \otimes \mathbb{C} \to T_{f(z)}\mathbb{R}^{2n} \otimes \mathbb{C}$$

with respect to the basis

$$\langle \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_m}, \frac{\partial}{\partial \overline{z}_1}, ..., \frac{\partial}{\partial \overline{z}_m} \rangle$$

and

$$\langle \frac{\partial}{\partial w_1}, ..., \frac{\partial}{\partial w_m}, \frac{\partial}{\partial \overline{w}_1}, ..., \frac{\partial}{\partial \overline{w}_m} \rangle$$

is then given by

$$\begin{pmatrix} \left(\frac{\partial f_i}{\partial z_j}\right)_{i,j} & \left(\frac{\partial f_i}{\partial \overline{z}_j}\right)_{i,j} \\ \left(\frac{\partial \overline{f}_i}{\partial z_j}\right)_{i,j} & \left(\frac{\partial \overline{f}_i}{\partial \overline{z}_j}\right)_{i,j} \end{pmatrix} \quad \text{and for } f \text{ holomorphic by} \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$$

In particular, for a holomorphic function f one has $\det J_{\mathbb{R}}(f)$. $\det\overline{J(f)}$=$|\det J(f)|^2$, which is non-negative.

In analogy to the implicit function theorem and the inverse functions theorem, for real functions, one has the following standard results.

> **Proposition 1.9 (Inverse function theorem)** *Let $F : U \to V$ be a holomorphic map between two open subsets $U, V \subset \mathbb{C}^n$. If $z \in U$ is regular, then there eixsts open subsets with $z \in U' \subset U$ and $f(z) \in V' \subset V$ such that f induces a biholomorphic map $f : U' \to V'$.*

As well as

> **Proposition 1.10 (Implicit function theorem)** *Let $U \subset C^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map where $m \geq n$. Suppose $z_0 \in U$ is a point such that*
> $$\det\left(\frac{\partial f_i}{\partial z_j}(z_0)\right)_{1\leq i,j\leq n} \neq 0.$$
> *Then there exists open subsets $U_1 \subset C^{m-n}$, $U_2 \subset \mathbb{C}^n$ and a holomorphic map $g : U_1 \to U_2$ such that $U_1 \times U_2 \subset U$ and $f(z) = f(z_0)$ if and only if*
> $$g(z_{n+1}, ..., z_m) = (z_1, ..., z_n)$$

Proof. Using the relation between the complex and real Jacobian explained above, one finds z is regular if and only if

$$\det J_{\mathbb{R}}(z) \neq 0.$$

That is, z is a regular point for the underlying real map. Thus the real inverse theorem applies and we are guaranteed the existence of a C^∞ inverse function

$$f^{-1} : V' \to U' \subset U$$

of f. It suffices to show f^{-1} satisfies the Cauchy-Riemann equations. Clearly,

$$\frac{\partial(f^{-1} \circ f)}{\partial z_j} = 0.$$

I.e., the identity map is holomorphic in each variable. And on the other hand f is holomorphic thus

$$\begin{aligned}\frac{\partial(f^{-1} \circ f)}{\partial z_j} &= \sum_{k=1}^{n} \frac{\partial f^{-1}}{\partial w_k} \cdot \frac{\partial f_k}{\partial \overline{z}_j} + \frac{\partial f^{-1}}{\partial \overline{w}_k} \cdot \frac{\partial \overline{f}_k}{\partial \overline{z}_j} \\ &= \sum_{k=1}^{n} \frac{\partial f^{-1}}{\partial \overline{w}_k} \cdot \frac{\partial \overline{f}_k}{\partial \overline{z}_j}\end{aligned}$$

Thus

$$\left(\frac{\partial f^{-1}}{\partial \overline{w}_k}\right)_k \overline{J(f)} = 0$$

And since $\det J_{\mathbb{R}}(f) \neq 0$ on U' we conclude that $(\frac{\partial f^{-1}}{\partial \overline{w}_k}) = 0$ on V' for all k, i.e., f^{-1} is holomorphic. We use the real version of the implicit function theorem to guarantee us the existence of the function g. We must now only show g is holomorphic. Clearly we have

$$\frac{\partial}{\partial \overline{z}_j}(f_i(g(z_{n+1},...,z_m),z_{n+1},...,z_m) = 0, \qquad \text{for } n+1 \leq j \leq m.$$

On the other hand holomorphicity of f gives us

$$\begin{aligned} \frac{\partial}{\partial \overline{z}_j}(f_i(g(z_{n+1},...,z_m),z_{n+1},...,z_m) &= \frac{\partial f_i}{\partial \overline{z}_j}(f_i(g(z_{n+1},...,z_m),z_{n+1},...,z_m) \\ &+ \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} \cdot \frac{\partial g_k}{\partial \overline{z}_j} + \frac{\partial f_i}{\partial \overline{z}_k} \cdot \frac{\partial \overline{g}_k}{\partial \overline{z}_j} \\ &= \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} \cdot \frac{\partial g_k}{\partial \overline{z}_j} \end{aligned}$$

Thus

$$\left(\frac{\partial f_i}{\partial z_j}\right)_{1\leq i,j\leq n} \cdot \left(\frac{\partial g_k}{\partial \overline{z}_j}\right)_{k=1,....,n} = 0, \qquad \text{for all } j.$$

This implies $\left(\frac{\partial g_k}{\partial \overline{z}_j}\right) = 0$ for $1 \leq k,j \leq n$. Hence the g_k are all holomorphic. □

Contrary to the real world, a holomorphic map is biholomorphic if and only if it is bijective. So the regularity of the Jacobian follows from the bijectvitiy. This is summarized in the following proposition.

Proposition 1.11 *Let $f : U \to V$ be a bijective holomorphic map between two open subsets $U, V \subset \mathbb{C}^n$. Then for all $z \in U$ one has $\det J(f)(z) \neq 0$. In particular, f is biholomorphic.*

Proof. The proof is done by induction. For $n = 1$, suppose f' has a zero. After a coordinate change we may assume

$$f(0) = f'(0) = 0.$$

Then the power series expansion of f has the form

$$f(z) = z^d h(z), \qquad \text{where } d \geq 2 \text{ and } h(0) \neq 0$$

In a small neighborhood of 0 we may consider the $d-$th root $\sqrt[d]{h(z)}$. Then

$$w := z^d \sqrt[d]{h(z)}$$

is a local coordinate. With respect to this coordinate f has the form

$$f(w) = w^d$$

thus f is non 1-1, a contradiction.

Now suppose n is arbitrary and assume the assertion is proven for all $k < n$. Let $z \in U$ such that $\det J(f)(z) = 0$, we will show this implies $J(f)(z) = 0$. Suppose that $\mathrm{rk} J(f)(z) = k \geq 1$. We may assume

$$\left(\frac{\partial f_i}{\partial z_j}(z) \right)_{1 \leq i,j \leq n}$$

is non-singular. I.e., has non-zero determinant. By the inverse function theorem,

$$\tilde{z}_i := f_i(z)$$

for $i = 1, ..., k$ and

$$\tilde{z}_i := z_i$$

for $i = k+1, ..., n$ form a local coordinate system around $z \in U$. Clearly f maps

$$U' := \{\tilde{z} : \tilde{z}_i = 0, \quad \text{for } i = 1, ..., k\} \cap U$$

bijectively onto

$$V' := \{w : w_i = 0, \quad \text{for } i = 1, ..., k\} \cap V.$$

But the restriction of f to U' is singular at z. This contradicts the induction hypothesis. Therefore $k = 0$. I.e., whenever $J(f)(z)$ is singular it vanishes completely.

Let $\det J(f)(z) = 0$ and assume z is a regular point of the holomorphic function

$$\det J(f) : U \to \mathbb{C}.$$

A neighborhood W of z in the pullback of the function $\det J(f)$ over $0 \in \mathbb{C}$ is biholomorphic to an open subset of C^{n-1}, by implicit function theorem, and for $n > 1$, this has positive dimension. This yields a function

$$f|_W : W \to \mathbb{C}^n$$

with vanishing Jacobian everywhere, which is therefore constant, this contradicts injectivity of f. It is left to show that there always exists a regular point of $\det J(f)$ in the fibre over 0.Let us consider the holomorphic function g. A priori, the fibre $Z(g) = g^{-1}(0)$ might not contain any $g-$ regular point. (e.g. $g = z^2$). Fix a point $z \in g^{-1}(0)$. For simplicity assume $z = 0$ and write

$$g = \prod_i g_i^{n_i}$$

where the g_i are holomorphic on $\mathbb{C}^n$ are coprime and irreducible. Hence

$$Z(g) = \bigcup Z(g_i^{n_i}) = \bigcup Z(g_i).$$

Note that two zero sets may intersect each other but according to the Corollary, they cannot be contained in each other for $i \neq j$. So $Z(g)$ might not contain any $g-$regular points, but every component $Z(g_i)$ contains g_i-regular points. So we wanted to show that the fibre of the holomorphic map

$$g = \det J(f) : U \to \mathbb{C}$$

contains a positive dimensional W. If there exists $g-$regular points in the fibre $g^{-1}(0)$, then W exists by the implicit function theorem. By above arguments, we see any component $Z(g_i)$, where $g = \prod_i g_i^{n_i}$ is the prime factor decomposition of g, contains such a set W as desired. □

Definition 1.12 By $\mathcal{O}_{\mathbb{C}^n}$ we denote the *sheaf of holomorphic functions on* $\mathbb{C}^n$. Thus, for any subset $U \subset \mathbb{C}^n$, the space of sections $\mathcal{O}_{\mathbb{C}^n}(U)$ of this sheaf over U is the set of all holomorphic functions $f : U \to \mathbb{C}$. The *stalk* $\mathcal{O}_{\mathbb{C}^n,z}$ of $\mathcal{O}_{\mathbb{C}^n}$ at a point $z \in \mathbb{C}^n$ is the set of all germs (U, f) where U is an arbitrary small open neighborhood of z and f is holomorphic on U.

Definition 1.13 Let R be an integral domain, i.e., R has no zero divisors. An element $f \in R$ is *irreducible* if it cannot be written as the product of two non-units in R. An integral domain is called a *unique factorization domain* (UFD) if every element can be written as a product of irreducibles, and if the factors are unique up to reordering and multiplication with units. This leads us to our next proposition.

Proposition 1.14 *The local ring* $\mathcal{O}_{\mathbb{C}^n,0}$ *is a UFD.*

Proof. The proof relies on one fact from Abstract Algebra, that is, if R is a UFD, then so is the polynomial ring $R[x]$. We prove this by induction on n. For the base case when $n = 1$, the ring

$$\mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}$$

is a field and thus a UFD. Suppose then that $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is a UFD. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$, we can choose coordinates such that the WPT can be applied. Thus

$$f = g \cdot h$$

where $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial and h is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$. As an element of $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$, g can be written uniquely as a product of irreducible elements $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. All that is left to show is that any irreducible factor g_i is also irreducible as an element in $\mathcal{O}_{\mathbb{C}^n,0}$.

Let us first show that any Weierstrass polynomial can be written as a product of irreducible Weierstrass polynomials, note not every irreducible factor g_i need be a Weierstrass polynomial. Let us assume a Weierstrass polynomial can be written as the product of non-units $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Consider the decomposition of g_i according to the WPT

$$g_i = \tilde{g}_i \cdot h_i$$

with Weierstrasss polynomials $\tilde{g}_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Note that since g is a Weierstrass polynomial, all the factors g_i are non-trivial on the z_1-line and thus satisfy the hypothesis of the Weierstrass prep theorem. Then

$$g = \prod \tilde{g}_i \cdot \prod h_i.$$

By uniqueness of the WPT,

$$g = \prod \tilde{g}_i, \quad \text{and} \quad \prod h_i = 1.$$

A priori, the $\tilde{g}_i$ need not be irreducible in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ as the h_i are just units in $\mathcal{O}_{\mathbb{C}^n,0}$. But since the degree of g as a polynomial in z_1 is finite, repeating the process leads to a decomposition, where either the $\tilde{g}_i$ are irreducible Weierstrass polynomials or elements in $\mathcal{O}_{\mathbb{C}^{n-1},0}$. To the latter, we can apply the induction hypothesis.

We conclude by showing any irreducible Weierstrass polynomial g is actually irreducible as an element of $\mathcal{O}_{\mathbb{C}^n,0}$. Suppose that

$$g = f_1 \cdot f_2$$

where $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,0}$ are non-units. We apply the WPT to both functions. Hence

$$f_i = g_i \cdot h_i, \quad \text{for } i = 1, 2.$$

And thus

$$g = (g_1 \cdot g_2) \cdot (h_1 \cdot h_2).$$

By the uniqueness part of the WPT, this gives us

$$g = g_1 \cdot g_2$$

which contradicts irreducibility of g as an element of $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and the theorem is proven. □

And important fact is that the ring of holomorphic functions in the origin is noetherian. This leads us to our next proposition.

Proposition 1.15 *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$, and let $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ be a Weierstrass polynomial of degree d. Then there exists $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that*

$$f = g \cdot h + r.$$

The functions h, r are uniquely determined.

Proof. For uniqueness, assume

$$f = g \cdot h_1 + r_1 = g \cdot h_2 + r_2.$$

Then

$$r_1 - r_2 = g(h_2 - h_1).$$

For $w = (z_2, ..., z_n)$ we consider the function

$$g_w(z_1) := g(z_1, z_2, ..., z_n).$$

Since g is a Weierstrass polynomial, g_w vanishes at the origin. Thus at $w = (0, ..., 0)$, g_w has a zero of order d. Thus for any small w, the polynomial g_w has d zeros (counted with multiplicities) close to 0. Then the same must hold for $(r_1 - r_2)_w$ which is a polynomial of degree $< d$, thus

$$(r_1 - r_2)_w \equiv 0$$

for a generic w and hence $r_1 = r_2$. To prove existence of h and r, we define h by

$$h(z_1, ..., z_n) := \frac{1}{2\pi i} \int_{|\xi|=\epsilon_1} \frac{f_w(\xi)}{g_w(\xi) \cdot (\xi - z_1)} d\xi,$$

where $|z_1| < \epsilon_1$. For ϵ_i, $i = 1, 2, ..., n$ small enough, we may assume

$$g_w(\xi) \neq 0, \qquad \text{for any } |\xi| = \epsilon_1 \text{ and } |z_j| < \epsilon_j \text{ for } j = 1, 2, ..., n$$

Thus by Lemma 1.2 h is holomorphic. All that is left to show is that

$$r := f - gh$$

is a polynomial in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ of degree $< d$. This is shown by the following computation

$$\begin{aligned}
r(z_1,...,z_n) &= \frac{1}{2\pi i}\int_{|\xi|=\epsilon_1}\frac{f_w(\xi)}{\xi - z_1}d\xi - \frac{g_w(z_1)}{2\pi i}\int_{|\xi|=\epsilon_1}\frac{f_w(\xi)}{g_w(\xi)\cdot(\xi - z_1)}d\xi \\
&= \frac{1}{2\pi i}\int_{|\xi|=\epsilon_1}\frac{f_w(\xi)\cdot(g_w(\xi)-g_w(z_1))}{g_w(\xi)\cdot(\xi - z_1)}d\xi \\
&= \frac{1}{2\pi i}\int_{|\xi|=\epsilon_1}\frac{f_w(\xi)}{g_w(\xi)}\cdot\left(\frac{(\xi^d - z_1^d) + \alpha_1(w)\cdot(\xi^{d-1}-z_1^{d-1}) + ...}{(\xi - z_1)}\right)d\xi \\
&= \frac{1}{2\pi i}\int_{|\xi|=\epsilon_1}\frac{f_w(\xi)}{g_w(\xi)}\cdot(z_1^{d-1}\beta_1(\xi,w) + z_1^{d-2}\beta_2(\xi,w) + ...)d\xi
\end{aligned}$$

where the $\alpha_j(w)$ are the coefficients of $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and the $\beta_j(\xi,w)$ are determined by them. The integrand in the last integral is a polynomial in z_1 of order $< d$ whose coefficients are holomorphic in (ξ,w). □

Next, we show the UFD $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian.

Proposition 1.16 *The local UFD $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian.*

Proof. We must show any ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ is finitely generated. We use induction on n again. For the base case $n = 1$ is trivial as any field is noetherian.

Next, we assume $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is noetherian. Then the polynomial ring $\mathcal{O}_{\mathbb{C}^n,0}[x]$ is noetherian as well. Let $I \subset \mathcal{O}_{\mathbb{C}^n}$ be a non-trivial ideal and choose $0 \neq f \in I$. Changing coordinates if necessary, we may assume the WPT can be applied. I.e.,

$$f = g \cdot h$$

where $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ a Weierstrass polynomial, and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ a unit. Hence $g \in I$. The ideal

$$I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$$

in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is generated by finitely many elements, namely, $g_1,...,g_k$.

For any other $\tilde{f} \in I$, the Weierstrass division theorem yields

$$\tilde{f} = g \cdot \tilde{h} + r, \qquad \text{for some } r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1].$$

And since $\tilde{f}, g \cdot \tilde{h} \in I$, this forces $r \in I$ as well. Therefore $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and thus

$$\tilde{f} = g \cdot \tilde{h} + \sum_{j=1}^{k} a_j \cdot g_j$$

This shows I is finitely generated by the elements $g_1,..,g_k$, and we are done by induction. , □

Our next proposition deal with functions vanishing on other functions.

Proposition 1.17 *Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be an irreducible function. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on $Z(g)$, then g divides f.*

Proof. By the WPT we may assume $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ is a Weierstrass polynomial of degree d. By the Weierstrass division theorem, one can find $h \in \mathcal{O}_{\mathbb{C}^n,0}$ and an $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ such that

$$f = g \cdot h + r.$$

By assumption, r_w vanishes on the zero set of g_w. If the zeros of g_w for generic w had multiplicity 1, then $r_w \equiv 0$, since r_w is of degree $< d$. It suffices to show the set of $w \in \mathbb{C}^{n-1}$ such that g_w has zeros with multiplicity > 1 is contained in the zero set of a non-trivial holomorphic function $\gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. Once we find γ, we conclude by using the fact that the complement of the zero set of γ is dense.

Since g is irreducible and $\frac{\partial g}{\partial z_1}$ is of degree $d-1$, there exists elements $h_1, h_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and $0 \neq \gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ such that

$$h_1 \cdot g + h_2 \cdot \frac{\partial g}{\partial z_1} = \gamma$$

since $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a UFD. If g_w has a zero ξ of multiplicity > 1, then

$$\gamma(\xi) = h_1(\xi, w) \cdot g_w(\xi) + h_2(\xi, w) \cdot \frac{\partial g}{\partial z_1}(\xi) = 0$$

□

By now the reader should be convinced that working in the local ring $\mathcal{O}_{\mathbb{C}^n,0}$ has many advantages when one is interested in only local properties of holomorphic functions. In some of the above arguments, we treated elements $f \in \mathcal{O}_{\mathbb{C}^n,0}$ as honest functions and associated to them their zero sets $Z(f)$.

Definition 1.18 The *germ* of a set in the origin $0 \in \mathbb{C}^n$ is given by a subset $X \subset \mathbb{C}^n$. Two subsets define the same germ if there exists an open neighborhood of $0 \in U \subset \mathbb{C}^n$ with

$$U \cap X = U \cap Y.$$

Sometimes one also writes $(X, 0)$ for a germ of a set in the origin. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$. Then denote by $Z(f)$ the germ of the zero set of f. I.e., if f is represented by a holomorphic function

$$f : U \to \mathbb{C},$$

then $Z(f)$ is represented by the zero set of this holomorphic function. Clearly, the germ $Z(f)$ does not depend on the chosen representative f. If f is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$ then $Z(f)$ is the empty set. Analagously, one defines

$$Z(f_1, ..., f_k) := Z(f_1) \cap ... \cap Z(f_k)$$

and more generally, $Z(A)$ as $\bigcap_{f \in A} Z(f)$, for a finite subset $A \subset \mathcal{O}_{\mathbb{C}^n,0}$.

Definition 1.19 A germ $X \subset \mathbb{C}^n$ in 0 is called *analytic* if there exists elements $f_1, ..., f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ such that X and $Z(f_1, ..., f_k)$ define the same germ.

Here is a global definition of this.

Definition 1.20 Let $U \subset \mathbb{C}^n$ be an open subset. An *analytic subset* of U is a closed subset

$X \subset U$ such that for any $x \in X$, there exists an open neighborhood of $x \in V \subset U$ and holomorphic functions $f_1, ..., f_k : V \to \mathbb{C}$ such that

$$X \cap V = \{z : f_1(z) = ... = f_k(z) = 0\}.$$

Definition 1.21 Let $X \subset \mathbb{C}^n$ be a germ in the origin. Then *I(X)* denotes the set of all elements $f \in \mathcal{O}_{\mathbb{C}^n,0}$ with $X \subset Z(f)$.

> **Lemma 1.21** *For any germ $X \subset \mathbb{C}^n$ the set $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is an ideal. If $(A) \subset \mathcal{O}_{\mathbb{C}^n,0}$ denotes the ideal generated by a subset $A \subset \mathcal{O}_{\mathbb{C}^n,0}$, then*
>
> $$Z(A) = Z((A))$$
>
> *and $Z(A)$ is analytic.*

This leads us to our next big Lemma.

> **Lemma 1.22** *If $X_1 \subset X_2$, then $I(X_2) \subset I(X_1)$. If $I_1 \subset I_2$, then $Z(I_2) \subset Z(I_1)$. For any analytic germ X one has*
>
> $$Z(I(X)) = X.$$
>
> *For any ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ one has $I \subset I(Z(I))$.*

Proof. The first two assertions are clear. Clearly we have that

$$X \subset Z(I(X)).$$

On the other hand there exists elements $f_1, ..., f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ with

$$X = Z(f_1, ..., f_k).$$

Then $f_1, ..., f_k \in I(X)$. Thus

$$Z(I(X)) \subset X = Z(f_1, ..., f_k).$$

Hence $X = Z(I(X))$. The last assertion is again trivial. □

Also note, for any two ideals $I, J \subset \mathcal{O}_{\mathbb{C}^n,0}$ one has

$$Z(I \cdot J) = Z(I) \cup Z(J)$$

and

$$Z(I + J) = Z(I) \cap Z(J).$$

Definition 1.23 An analytic germ is *irreducible* if the following condition is satisfied: Let $X = X_1 \cup X_2$, where X_1, X_2 are analytic germs. Then either $X = X_1$ or $X = X_2$. This property translates easily into an algebraic property of the associated ideal.

> **Lemma 1.24** *An analytic germ X is irreducible if and only if $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is a prime ideal.*

Proof. First assume X is irreducible. Suppose $f_1 \cdot f_2 \in I(X)$. Then

$$X = (X \cap Z(f_1)) \cup (X \cap Z(f_2)).$$

Thus

$$X = X \cap Z(f_j), \qquad \text{for } j = 1 \text{ or } j = 2,$$

hence at least one of the functions f_1 or f_2 vanishes on X and is therefore contained in the ideal $I(X)$.

On the other hand suppose $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is a prime ideal. Let

$$X = X_1 \cup X_2$$

with X_j analytic for $j = 1, 2$. If $f_j \in I(X_j)$, $j = 1, 2$, then $f_1 \cdot f_2 \in I(X)$. Hence $f_1 \in I(X)$ or $f_2 \in I(X)$. Thus it suffices to show that if $X \neq X_1$ and $X \neq X_2$, then there exists elements $f_1 \in I(X_1) \setminus I(X)$ and $f_2 \in I(X_2) \setminus I(X)$ however, this follows by Lemma 1.22. □

Lemma 1.24 also shows that for $f \in \mathcal{O}_{\mathbb{C}^n,0}$ the zero set $Z(f)$ is irreducible if and only if there exists an irreducible $g \in \mathcal{O}_{\mathbb{C}^n,0}$ such that

$$f = g^k, \qquad \text{for some } k.$$

Indeed, if $f = g^k$ and g is irreducible, then $Z(f) = Z(g)$, and if $h \in I(Z(f))$ then g divides h by Proposition 1.17. This yields

$$I(Z(g)) = (g)$$

and thus $Z(g)$ is irreducible. Conversely, if

$$f = \prod g_i^{n_i},$$

then

$$Z(f) = \bigcup Z(g_i)$$

which cannot be irreducible except for the case when $f = g^k$ with g irreducible.

Definition 1.25 Recall the *radical* of an ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ is the ideal $\sqrt{I}$ of all elements $f \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f^k \in I$ for some $k > 0$. One can easily prove $\sqrt{I} \subset I(Z(I))$. But the reverse containment is in the following theorem.

Proposition 1.26 *If $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ is any ideal, then $\sqrt{I} = I(Z(I))$.*

The proof relies on the following theorem:

Theorem 1.27 *Let $X \subset \mathbb{C}^n$ be an irreducible analytic germ defined by a prime ideal $\mathcal{P} \subset \mathcal{O}_{\mathbb{C}^n,0}$. Then one can find a coordinate system*

$$(z_1, z_2, ..., z_{n-d}, z_{n-d+1}, ..., z_n)$$

such that the projection

$$(z_1, ..., z_{n-d}) \mapsto (z_{n-d+1}, ..., z_n)$$

induces a surjective map of germs

$$\pi : X \to \mathbb{C}^d$$

and such that the induced ring homomorphism

$$\mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{P}$$

is a finite integral ring extension.

We presume with the proof of Proposition 1.26.

Proof. We have seen it suffices to prove the assertion holds for principal prime ideals. Since $\mathcal{P} \subset I(Z(\mathcal{P}))$, it suffices to show any $f \in I(Z(\mathcal{P}))$ is in $\mathcal{P}$. For an appropriate coordinate system $(z_1, ..., z_n)$, the induced element $\tilde{f} \in \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{P}$satisfies an irreducible algebraic equation

$$\tilde{f}^k + a_1 \tilde{f}^{k-1} + ... + a_k = 0,$$

over $\mathcal{O}_{\mathbb{C}^d,0}$. I.e., $a_i = a_i(w) \in \mathcal{O}_{\mathbb{C}^d,0}$ where $w = (z_{n-d+1}, ..., z_n)$. Since f vanishes along $Z(\mathcal{P})$, the 0th coordinate a_k does as well. As $Z(\mathcal{P}) \to \mathbb{C}^d$ is surjective, we have that $a_k = 0$. Hence, the above algebraic expression cannot be irreducible except for $k = 1$. Therefore $\tilde{f} = 0$ and thus $f \in \mathcal{P}$, as desired. ☐

Definition: Let X be an irreducible analytic germ defined by a prime ideal $\mathcal{P} \subset \mathcal{O}_{\mathbb{C}^n,0}$. Then the *dimension* of X is defined by dim(X)=d where d is as in Theorem 1.27.

Definition: Let $U \subset \mathbb{C}^n$ be open. A *meromorphic* function f on U is a function on the complement of a nowhere dense subset $S \subset U$ with the following property: There exists an open cover $U = \bigcup U_j$ and holomorphic functions

$$g_j, h_j : U_j \to \mathbb{C}$$

with

$$h_j|_{U_j \setminus S} \cdot f|_{U_j \setminus S} = g_j|_{U_j \setminus S}.$$

Let $K(U)$ denote the set of all meromorphic functions defined on some U. One can easily verify that $K(U)$ is a field if U is connected. Let f be meromorphic on U. That is, $f \in K(X)$. Then, for any $z \in U$ the meromorphic function f in a neighborhood of z is given by $\frac{g}{h}$ with $g, h \in \mathcal{O}_{\mathbb{C}^n,z}$. If we choose g, h to be coprime, then they are unique up to units. Hence the zero set and the pole set of a meromorphic function are well-defined.

Definition: Let f be a meromorphic function on an open subset $U \subset \mathbb{C}^n$. Then the *zero set* $Z(f) \subset U$ of f and the *pole set* $P(f) \subset U$ are the analytic sets that in every point $z \in U$ are given by $Z(g)$ and $Z(h)$ respectively, where f on an open neighborhood of z is given by $\frac{g}{h}$ with $g, h \in \mathcal{O}_{\mathbb{C}^n,0}$ coprime.

> **Proposition 1.28** *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be irreducible. Then for sufficiently small ϵ and $z \in B_\epsilon(0)$ the induced element $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is irreducible. If $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ are coprime, then they are corprime on $\mathcal{O}_{\mathbb{C}^n,0}$ for z in a sufficiently small neighborhood of* 0.

Proof. We may assume $f \in \mathcal{O}_{\mathbb{C}^n,0}[z_1]$ is a Weierstrass polynomial. Suppose that f as an element of $\mathcal{O}_{\mathbb{C}^n,z}$ is reducible. Then

$$f = f_1 \cdot f_2$$

with $f_j \in \mathcal{O}_{\mathbb{C}^n,z}$ non-units for $j = 1, 2$. That is,

$$f_1(z) = f_2(z) = 0.$$

Thus

$$\frac{\partial f}{\partial z_1}(z) = \frac{\partial f_1}{\partial z_1}(z) \cdot f_2(z) + f_1(z) \cdot \frac{\partial f_2}{\partial z_1}(z) = 0.$$

Thus the set of points $z \in B_\epsilon(0)$ where f as an element of $\mathcal{O}_{\mathbb{C}^n,z}$ is reducible is contained in the analytic set $Z(f, \frac{\partial f}{\partial z_1})$. We must show this is a proper subset of $Z(f)$. If not, then $\frac{\partial f}{\partial z_1}$ would vanish on $Z(f)$. Since f is irreducible in $\mathcal{O}_{\mathbb{C}^n,0}$, we can apply Proposition 1.17. This yields a contradiction for degree reasons.

For the latter assertion, we may assume $f, g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ are Weierstrass polynomials. then f and g are coprime in $\mathcal{O}_{\mathbb{C}^n,0}$ if and only if they are coprime in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. By the Gauss Lemma, the polynomials f and g are coprime if and only if there exists polynomials $h_1, h_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ such that

$$0 \neq \gamma = h_1 \cdot f + h_2 \cdot g \in \mathcal{O}_{\mathbb{C}^{n-1},0}.$$

This immediately proves the assertion where the open neighborhood of the origin is given by the open subsets where γ, h_1, h_2, f and g are defined and γ does not vanish. □

The next proposition will be used later in the proof of Siegel's theorem which says the transcendence degree of the function field of a compact complex manifold is at most the dimension of the manifold.

Proposition 1.29 *Let $\epsilon := (\delta, ..., \delta)$ and let f be a holomorphic function on an open neighbourhood of the closure of the polydisk $\overline{B_\epsilon(0)}$. Assume that f vanishes of order k at the origin. I.e., in the power series expansion, non-trivial monomials of degree $< k$ do not occur.*
If $|f(z)|$ for $z \in \overline{B_\epsilon(0)}$ can be bounded from above by C, then

$$|f(z)| \leq C \left(\frac{|z|}{\delta} \right)^k,$$

for all $z \in \overline{B_\epsilon(0)}$

Proof. Fix $0 \neq z \in \overline{B_\epsilon(0)}$ and define a holomorphic function g_z of one variable as follows: For $w \leq \delta$ one sets

$$g_z(w) := w^{-k} f\left(w \cdot \frac{z}{|z|} \right).$$

Then

$$|g_z(w)| \leq \delta^{-k} C \qquad \text{for } |w| = \delta.$$

The maximum principle implies that

$$|g_z(w)| \leq \delta^{-k} C \qquad \text{for } |w| \leq \delta.$$

Hence

$$|z|^{-k} |f(z)| = |g_z(|z|)| \leq \delta^{-k} C.$$

And the proposition is proven. □

Exercise 1 Show that every holomorphic map $f : \mathbb{C} \to \mathbb{H} := \{z : \text{Im}(z) > 0\}$ is constant.

Proof. Consider the function

$$g(z) := e^{if(z)}$$

Then $g(z)$ is bounded and thus by Liouville's Theorem since $g(z)$ is entire as well, $g(z)$ is constant forcing $f(z)$ to be constant. □

Exercise 2 Show the real and imaginary parts of a holomorphic function with $f = u + iv$ are harmonic. I.e.,

$$\sum_j \frac{\partial^2 u}{\partial x_j^2} + \sum_j \frac{\partial^2 u}{\partial y_j^2} = 0$$

and similarly for v.

Proof. Since f is holomorphic, we have that u, v satisfy the Cauchy-Riemann equations. I.e.,

$$u_x = v_y, u_y = -v_x$$

Thus differentiating once more we obtain

$$u_{xx} = v_{yx}, u_{yy} = -v_{xy}.$$

By Schwarz's Theorem

$$v_{xy} = v_{yx}$$

and thus

$$u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0,$$

similarly for v. □

Exercise 3

(Text not finished yet, work in progress for a while. This is my first attempt at a text book or informal notes on a given topic).

2. Complex and Hermitian structures

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