Math 560 - Real Analysis I Notes
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Contents

§ 1. Rudiments of $\mathbb{R}^d$

In undergraduate analysis we learned about the Riemann integral which limits the class of functions one can integrate over. We extend this notion of "length" in 1-dimension, area in 2, and volume in 3-space into subsets E of $\mathbb{R}^d$. We begin with a point $x \in \mathbb{R}^d$ we denote by

$$x = (x_1, ..., x_d), \qquad x_i \in \mathbb{R}, i = 1, 2, ..., d.$$

Here addition is defined point-wise, that is, for any $x, y \in \mathbb{R}^d$, one has

$$x + y := (x_1 + y_1, ..., x_d + y_d).$$

Similarly, for any scalar $\delta \in \mathbb{R}$, one has

$$\delta x := (\delta x_1, ..., \delta x_d).$$

Define the **norm** of $x \in \mathbb{R}^d$ to be $|x|$ defined via

$$|x| := (x_1^2 + ... + x_d^2)^{\frac{1}{2}}.$$

Thus the "distance" between $x, y \in \mathbb{R}^d$ is given via

$$|x - y|.$$

For a given set $E \subseteq \mathbb{R}^d$ define its **complement** as E^c via

$$E^c := \{x \in \mathbb{R}^d : x \in \mathbb{R}^d \bigwedge x \notin E\}.$$

For two given set $E, F \subseteq \mathbb{R}^d$, define their **distance** to be given via

$$d(E, F) := \inf |x - y|.$$

Here the infimum is taken over all $x \in E, y \in F$.

§ 1.1 Elements of the Topology

Define the **open ball** of radius r centered at x via

$$B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}.$$

A subset $E \subseteq \mathbb{R}^d$ is defined to be **open** if for every $x \in E$, there exists and $r > 0$ such that

$$B_r(x) \subseteq E.$$

Then define a set $F \subseteq \mathbb{R}^d$ to be **closed** if F^c is open. It is worthy to note that the open subsets of $\mathbb{R}^d$ are elements of the topology, $\tau_{\mathbb{R}^d}$, on $\mathbb{R}^d$. Furthermore, note that any arbitrary union of open sets is open while any finite union of closed sets is closed. Additionally, arbitrary intersections of closed need be closed while arbitrary intersections of open need not be open. More formally speaking, if E_α is open for every $\alpha \in A$, some indexing set, then

$$\bigcup_\alpha E_\alpha$$

is open. A subset $E \subseteq \mathbb{R}^d$ is **bounded** if it is contained in some finite radius ball. A bounded set is **compact** if it is also closed. In $\mathbb{R}^d$, compact is equivalent to closed and bounded. This is more famously known as the Heine-Borel Theorem. A more "topological" definition of compact sets is given any open covering of E, that is, for a given collection $\{\mathcal{O}_\alpha : \alpha \in A\}$ of open subsets of $\mathbb{R}^d$ with the property that

$$E \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha,$$

there will always exist some finite subset $B \subseteq A$ such that

$$E \subseteq \bigcup_{\alpha \in B} \mathcal{O}_\alpha.$$

That is, any (arbitrary) open covering of E contains a finite sub-covering of E. Define $x \in \mathbb{R}^d$ to be a **limit point** of some $E \subseteq \mathbb{R}^d$ if for every $r > 0$, the intersection

$$B_r(x) \setminus \{x\} \bigcap E$$

is non-trivial. That is, $B_r(x) \setminus \{x\}$ contains points of E. Well, at least one point. A point $x \in \mathbb{R}^d$ is said to be an **isolated point** if $x \in E$ and there exists an $r > 0$ such that

$$B_r(x) \bigcap E = \{x\}.$$

A point $x \in \mathbb{R}^d$ is said to be an **interior point** of $E \subseteq \mathbb{R}^d$ if there exists an $r > 0$ such that

$$B_r(x) \subseteq E.$$

The set of all interior points of a given set $E \subseteq \mathbb{R}^d$ is denoted $\mathrm{int}(E)$. We denote the **closure** of E via $\overline{E}$ which consists of E union with all of its limit points. Let the **boundary** of E be denoted via ∂E as

$$\partial E := \{x \in \mathbb{R}^d : x \in \overline{E} \setminus \mathrm{int}(E)\}.$$

Not that the closure of any set is a closed set and that a set is closed if and only if it contains all of its own limit points. Lastly, a set $E \subseteq \mathbb{R}^d$ is **perfect** if E contains no isolated points.

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§ 1.2 Rectangles

Define a (closed) **rectangle** R in $\mathbb{R}^d$ as the product of d $1-$dimensional closed and bounded intervals. That is,

$$R = \prod_{k=1}^{d} [a_k, b_k].$$

Thus it makes sense for us to define the "length" of these intervals as

$$b_1 - a_1, ..., b_d - a_d.$$

Then the **volume** of R, denoted $|R|$ via

$$|R| := \prod_{k=1}^{d} (b_k - a_k).$$

Similarly one can define an **open** rectangle as

$$R = \prod_{k=1}^{d} (a_k, b_k).$$

Furthermore, a rectangle is a **cube** if one has

$$b_m - a_m = b_n - a_n$$

For every $m, n \in \{1, 2, ..., d\}$. Thus if $Q \subseteq \mathbb{R}^d$ is a cube with side length l, one has that

$$|Q| = l^d.$$

A union of rectangles is said to be **almost disjoint** if the interiors are disjoint. This leads us to our first result.

> **Lemma 1:** *If a rectangle is the almost disjoint union of finitely many other rectangles say $R = \bigcup_{k=1}^{N} R_k$, then*
>
> $$|R| = \sum_{k=1}^{N} |R_k|.$$

Proof. We extend the sides of the $R_1, ..., R_N$ indefinitely. This construction yields finitely many rectangles $\overline{R}_1, ..., \overline{R}_M$ and a partition $J_1, ..., J_N$ of integers between 1 and M such that the unions

$$R = \bigcup_{j=1}^{M} \overline{R}_j$$

and

$$R_k = \bigcup_{j \in J_k} \overline{R}_j$$

are almost disjoint. For the rectangle R, we see that

$$|R| = \sum_{j=1}^{M} |\overline{R}_j|.$$

This is because our grid partitions the sides of R and each $\overline{R}_j$ consists of taking products of the intervals in these partitions. Thus when we add the volumes of the $\overline{R}_j$ we sum up the corresponding products of the lengths of these intervals that arise which holds for the $R_1, ..., R_N$. Thus we get

$$\begin{aligned}
|R| &= \sum_{j=1}^{M} \overline{R}_j \\
&= \sum_{k=1}^{N} \sum_{j \in J_k} |\overline{R}_j| \\
&= \sum_{k=1}^{N} |R_k|.
\end{aligned}$$

as needed. □

Furthermore we have the result

Lemma 2: *If $R_1, ..., R_N$ are rectangles and $R \subset \bigcup_{k=1}^{N} R_k$, then*

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Proof. By proof above, note the sets corresponding to the J_k need not be disjoint anymore. □

This leads us to a big theorem.

Theorem 3 *Every open subset $\mathcal{O} \subseteq \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Proof. For each $x \in \mathcal{O}$, let I_x denote the largest interval containing x such that

$$I_x \subset \mathcal{O}.$$

Moreover since $\mathcal{O}$ is open, there exists some $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset \mathcal{O}.$$

So if

$$a_x := \inf\{a < x : (a, x) \subset \mathcal{O}\}$$

and

$$b_x := \sup\{x < b : (x, b) \subset \mathcal{O}\}$$

we then have that

$$a_x < x < b_x.$$

Note that there may exists infinitely many values for a_x, b_x. If we now take

$$I_x := (a_x, b_x),$$

then $x \in I_x$ by construction and we even have

$$I_x \subset \mathcal{O}.$$

Thus

$$\mathcal{O} := \bigcup_{x\in\mathcal{O}} I_x.$$

Suppose that for $x, y \in \mathbb{R}$ we have that I_x, I_y intersect. Then

$$x \in I_x \cup I_y \subset \mathcal{O}.$$

By maximality of the intervals we have constructed, we have that

$$I_x \cup I_y \subset I_x$$

and

$$I_x \cup I_y \subset I_y.$$

Forcing

$$I_x = I_y.$$

Thus two distinct intervals in

$$\mathcal{I} := \{I_x\}_{x\in\mathcal{O}}$$

need be disjoint. We must now only show there are countably many distinct intervals in $\mathcal{I}$. By their disjointness and by density, each I_x will always contain a rational number. As the intervals are distinct, they contain distinct rationals. This forces $\mathcal{I}$ to be countable as desired. □

Typically, if $\mathcal{O}$ is open with

$$\mathcal{O} = \bigcup_{j=1}^{\infty} I_j,$$

where the I_j are disjoint open intervals, the measure of $\mathcal{O}$ ought to be

$$\sum_{j=1}^{\infty} |I_j|.$$

Now we generalize the previous theorem to all of $\mathbb{R}^d$:

> **Theorem 4** *Every open subset $\mathcal{O} \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.*

Proof. By construction we will find a countable collections $\mathcal{Q}$ of closed cubes whose interiors are disjoint. We would like to have

$$\mathcal{O} = \bigcup_{Q\in\mathcal{Q}} Q.$$

First consider the grid in $\mathbb{R}^d$ by taking all closed cubes of unit length with integer vertices. I.e., consider the grid generated by the lattice $\mathbb{Z}^d$. We also use grids formed by cubes of side length 2^{-N} obtained by successively bisecting the original grid in half. We keep or toss our cubes by the following rule: If

$$Q \subset \mathcal{O}$$

then we accept Q and if Q intersects both $\mathcal{O}$ and $\mathcal{O}^c$ then we accept it tentatively, and if

$$Q \subset \mathcal{O}^c$$

then we toss it out. As a second step we bisect the tentative ones into 2^d cubes of side length $1/2$. We repeat this process. By construction, we have that this collection $\mathcal{Q}$ is not only countable but consists of almost disjoint cubes. To see why

$$\mathcal{O} = \bigcup_{Q \in \mathcal{Q}} Q,$$

for any $x \in \mathcal{O}$ there exists a cube with side length 2^{-N} containing x properly contained inside of $\mathcal{O}$. This cube has either been accepted or is contained in a cube previously accepted in another stage. This shows the union covers $\mathcal{O}$. $\square$

Thus if $\mathcal{O} = \bigcup_{j=1}^{\infty} R_j$, then it is natural to assign to $\mathcal{O}$ the measure of

$$\sum_{j=1}^{\infty} R_j.$$

The next section will be of great significance as it shows an interesting set of measure 0.

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§ 1.3 Cantors Set

We define what is know as the **Cantor Set**. We begin with defining the closed unit interval as

$$\mathcal{C}_0 = [0, 1].$$

Then take $\mathcal{C}_1$ to be the set obtained from cutting out the middle third from $\mathcal{C}_0$. Thus we get

$$\mathcal{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

Repeating this process gives us

$$\mathcal{C}_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

We repeat this process indefinitely which gives us a sequence $\mathcal{C}_k$ of decreasing compact sets in the sense that

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset ... \supset \mathcal{C}_k \supset \mathcal{C}_{k+1} \supset ...$$

The Cantor set is then defined to be the intersection

$$\mathcal{C} = \bigcap_{k=0}^{\infty} \mathcal{C}_k.$$

Clearly $\mathcal{C}$ is non-empty as all end points of the $\mathcal{C}_k$ belong to $\mathcal{C}$. Note that $\mathcal{C} \subset \mathbb{R}$ is both closed and bounded and thus compact. Furthermore the space is totally disconnected with no isolated points.

Naturally one would like to know the size or measure of this Cantor Set. Not to be confused with the cardinality of the set which is the same as the continuum. In terms of measure the Cantor Set is rather small and in fact has measure 0. As $\mathcal{C}_k$ is a disjoint union of 2^k intervals each of length 3^{-k}, then we have that the length of each of the $\mathcal{C}_k$ is just $(\frac{2}{3})^k$ which tends to 0 as $k \to \infty$. It is worth noting that the *Hausdorff dimension* of the cantor set is

$$\frac{\log 2}{\log 3}.$$

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§ 2. The Exterior Measure

The exterior measure attempts to describe the size of a set using outside approximations. We make this definition precise as follows. If E is *any* subset of $\mathbb{R}^d$, define the **exterior measure** of E as

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|.$$

Here the infimum is taken over all countable coverings

$$E \subseteq \bigcup_{j=1}^{\infty} Q_j$$

by closed cubes. Note $m_*(E) \in [0, \infty]$. We calculate the exterior measure of a few basic examples.

Example 1 The exterior measure of a point is just zero. Furthermore, the exterior measure of a collection of points is also zero, even countably infinite. Take $\mathbb{Q}$ for example. We also note the exterior measure of the empty set is 0.

Example 2 The exterior measure of closed cubes is equal to its volume. To see this, let Q be a closed cube in $\mathbb{R}^d$. Then since Q covers itself we have

$$m_*(Q) \leq |Q|.$$

We must show then that

$$|Q| \leq m_*(Q),$$

which would give us equality. Fix $\epsilon > 0$ and for each j, choose an open cube U_j containing Q_j such that

$$|U_j| \leq (1 + \epsilon)|Q_j|.$$

As $\bigcup_j U_j$ is a covering of $Q = \bigcup_j Q_j$ which is compact thus we can find a finite sub-covering. That is, we can find a finite subset $B \subset \mathbb{N}$ such that

$$Q \subseteq \bigcup_{j \in B} U_j.$$

Consider the closures

$$\{\overline{U_j}\}_{j\in B}.$$

Then by Lemma 2, since $Q \subset \bigcup_{j\in B} U_j$ we have that (since closure of an open set has the same measure)

$$|Q| \leq \sum_{j\in B} |U_j|.$$

Then we have

$$\begin{aligned} |Q| &\leq \sum_{j\in B} |U_j| \\ &\leq \sum_{j\in B} (1+\epsilon)|Q_j| \\ &= (1+\epsilon)\sum_{j\in B} |Q_j| \\ &\leq (1+\epsilon)\sum_{j\in \mathbb{N}} |Q_j| \\ &= (1+\epsilon)m_*(Q). \end{aligned}$$

as ϵ was arbitrary we are done.

Example 3 If Q is an open cube, then we still have that $m_*(Q) = |Q|$. Since $Q \subseteq \overline{Q}$ and their volumes agree, we have

$$m_*(Q) \leq |Q|.$$

To show the reverse inequality, if Q_0 is a closed cube inside of Q then any (countable) covering of Q also covers Q_0. Hence

$$|Q_0| \leq m_*(Q).$$

We can pick Q_0 with volume arbitrarily close to the value of $|Q|$, we get

$$|Q| \leq m_*(Q).$$

We then have equality as needed.

Example 4 The exterior measure of a rectangle agrees with its volume. By example 2,

$$|R| \leq m_*(R).$$

To get the reverse inequality, consider the collection $\mathcal{Q}$ of all cubes properly contained inside of R and the collection $\mathcal{Q}'$ of all cubes intersecting R^c. Then

$$R \subseteq \bigcup_{Q\in\mathcal{Q}\cup\mathcal{Q}'} Q.$$

As $\bigcup_{Q\in\mathcal{Q}} Q \subseteq R$ one gets

$$\sum_{Q\in\mathcal{Q}} |Q| \leq |R|.$$

Moreover there are $\mathcal{O}(k^{d-1})$ cubes in $\mathcal{Q}'$, with volume k^{-d}, so that

$$\sum_{Q\in\mathcal{Q}'}|Q| = \mathcal{O}(\frac{1}{k})$$

thus we get

$$\sum_{Q\in\mathcal{Q}\cup\mathcal{Q}'}|Q| \leq |R| + \mathcal{O}(\frac{1}{k}).$$

Sending $k\to\infty$ we get the desired result.

Example 5 Note $m_*(\mathbb{R}^d) = \infty$. Since any covering of $\mathbb{R}^d$ also covers any cube $Q\subset\mathbb{R}^d$. Thus we get

$$|Q| \leq m_*(\mathbb{R}^d).$$

Since Q can have arbitrary large volume we conclude that $m_*(\mathbb{R}^d) = \infty$.

Example 6 The cantor set $\mathcal{C}$ has exterior measure zero. To see this, note by construction we have $\mathcal{C}\subset\mathcal{C}_k$ for every k. Each $\mathcal{C}_k$ has 2^k disjoint intervals each of length 3^{-k}. Thus

$$0 \leq m_*(\mathcal{C}) \leq \left(\frac{2}{3}\right)^k.$$

This goes to 0 as $k\to\infty$.

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§ 2.2 Properties of m_*

We note some of the key properties of the exterior measure. First, given a subset $E\subset\mathbb{R}^d$, for any $\epsilon>0$, there will exists a covering $\{Q_j\}_j$ of E, i.e.,

$$E\subset\bigcup_j Q_j,$$

such that

$$\sum_j m_*(Q_j) \leq m_*(E) + \epsilon.$$

That is, E can be approximated from the outside by closed cubes arbitrarily close to E.

Observation 1 (Monotonicity) *If $E_1\subset E_2$, then*

$$m_*(E_1)\leq m_*(E_2).$$

Proof. If $\{Q_j\}$ is any covering of E_2 by closed cubes, then it is also a covering for E_1 and thus the infimum for E_1 is taken over a larger collection than that of E_2. □

Observation 2 (Countable sub-additivity) *If $E=\bigcup_j E_j$, then*

$$m_*(E)\leq\sum_j m_*(E_j).$$

Proof. We may assume for every j that

$$m_*(E_j) < \infty.$$

Otherwise we get equality. Let $\epsilon > 0$ be given. For each j one is then guaranteed of a collection $\{Q_{k,j}\}_{k\in\mathbb{N}}$ of closed cubes with

$$\sum_{k\in\mathbb{N}} |Q_{k,j}| \leq m_*(E_j) + \frac{\epsilon}{2^j}.$$

Then $E \subset \bigcup_{j,k\in\mathbb{N}} Q_{k,j}$ is a covering of E by closed cubes. Thus we get that

$$\begin{aligned} m_*(E) &\leq \sum_{j,k} |Q_{k,j}| \\ &= \sum_j \sum_k |Q_{k,j}| \\ &\leq \sum_j (m_*(E_j) + \frac{\epsilon}{2^j}) \\ &= \sum_j m_*(E_j) + \epsilon. \end{aligned}$$

We are finished as this holds for any given $\epsilon > 0$. □

Observation 3 (Approximation by open sets) *If* $E \subset \mathbb{R}^d$, *then* $m_*(E) = \inf m_*(\mathcal{O})$. Here the infimum is taken over all open set $\mathcal{O}$ containing E.

Proof. Let $\mathcal{O}$ be an open cover for E. Then by monotonicity we get

$$m_(E) \leq \inf m_*(\mathcal{O}).$$

To see the reverse inequality, let $\epsilon > 0$ be given. We choose our cubes Q_j such that $E \subset \bigcup_j Q_j$, such that

$$\sum_j |Q_j| \leq m_*(E) + \frac{\epsilon}{2}.$$

For each j, let Q_j^0 denote the open cube containing Q_j such that

$$|Q_j^0| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}.$$

Then $\mathcal{O} = \bigcup_j Q_j^0$ is an open set thus by monotonicity, we get

$$\begin{aligned} m_*(\mathcal{O}) &\leq \sum_j m_*(Q_j^0) \\ &= \sum_j |Q_j^0| \\ &\leq \sum_j \left(|Q_j| + \frac{\epsilon}{2^{j+1}} \right) \\ &\leq \sum_j |Q_j| + \frac{\epsilon}{2} \\ &\leq m_*(E) + \epsilon. \end{aligned}$$

Hence $\inf m_*(\mathcal{O}) \leq m_*(E)$. as needed. $\square$

Observation 4 *If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then*

$$m_*(E) = m_*(E_1) + m_*(E_2).$$

Proof. By monotonicity, we get that

$$m_*(E) \leq m_*(E_1) + m_*(E_2).$$

We must show the reverse inequality. Choose a $\delta > 0$ such that

$$d(E_1, E_2) > \delta > 0.$$

For $\epsilon > 0$, select a covering of E by closed cubes,

$$E \subset \bigcup_j Q_j$$

such that

$$\sum_j |Q_j| \leq m_*(E) + \epsilon.$$

After subdividing the Q_j we can assume they each have diameter less than δ. Then by our choice of δ, these newly subdivided Q_j intersect either E_1 or E_2. Let J_1, J_2 denote the set of indices j for which Q_j intersects E_1, E_2 respectively. Then we get that

$$J_1 \cap J_2 = \emptyset$$

and that

$$E_1 \subset \bigcup_{j \in J_1} Q_j, E_2 \subset \bigcup_{j \in J_2} Q_j.$$

Thus we have

$$\begin{aligned} m_*(E_1) + m_*(E_2) &\leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \\ &\leq \sum_j |Q_j| \\ &\leq m_*(E) + \epsilon. \end{aligned}$$

We are finished as ϵ was arbitrary. $\square$

Observation 5 *If E is a contable union of almoast disjoint cubes, then*

$$m_*(E) = \sum_j |Q_j|.$$

Proof. For some fixed $\epsilon > 0$ let $\tilde{Q}_j$ denote the cube properly contained inside of Q_j such that

$$|Q_j| \leq |\tilde{Q}_j| + \frac{\epsilon}{2^j}.$$

Then for every N, the cubes

$$\tilde{Q}_1, \tilde{Q}_2, ..., \tilde{Q}_N$$

are disjoint thus have some finite distance between them. Thus we can apply a repeated use of our previous observation to see that

$$\begin{aligned} m_*(\bigcup_j \tilde{Q}_j) &= \sum_{j=1}^{N} |\tilde{Q}_j| \\ &\geq \sum_{j=1}^{N} \left(|Q_j| - \frac{\epsilon}{2^j} \right) \end{aligned}$$

Since $\sum_{j=1}^{N} Q_j \subset E$ we have that

$$m_*(E) \geq \sum_{j=1}^{N} |Q_j| - \epsilon.$$

As $N \to \infty$, we see that

$$\sum_j |Q_j| \leq m_*(E) + \epsilon.$$

This holds for any arbitrary $\epsilon > 0$ thus together with Observation 2, we get equality as desired. □

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§ 3 The Lebesgue measure

There are a number of ways one can define measurability which all turn out to be the same.

Definition: We say that a subset $E \subset \mathbb{R}^d$ is *Lebesgue measurable* or simply *measurable* if for any $\epsilon > 0$, there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ such that

$$m_*(\mathcal{O} \setminus E) \leq \epsilon.$$

This can be compared to Observation 3 which holds for *all* sets. If E is measurable we define its *Lebesgue measure* $m(E)$ by

$$m_*(E) := m(E).$$

Similarly, for any subset $E \subseteq \mathbb{R}^d$ and any given set A, E is (Lebesgue) measurable if one has

$$m(A) = m(A \cap E) + m(A \cap E^c).$$

The Lebesgue measure inherits all Observations of the exterior measure.

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§ 3.1 Properties of m

We immediately inherit a few key properties of the Lebesgue measure.

Property 1 *Every open set in $\mathbb{R}^d$ is measurable.*

Proof. Let $\mathcal{O} \subseteq \mathbb{R}^d$ be open. Then by Theorem 3, one can write $\mathcal{O}$ as a disjoint union of open intervals. That is,

$$\mathcal{O} = \prod_{k=1}^{d} (a_k, b_k).$$

Where each interval is measurable and furthermore a countable union of measurable sets is still measurable thus $\mathcal{O}$ is measurable as needed. □

Property 2 *If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable itself.*

Proof. To show F is measurable, let $\epsilon > 0$ be fixed. Then there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ with

$$m_*(\mathcal{O}) \leq \epsilon.$$

Since $\mathcal{O} \setminus E \subset \mathcal{O}$, we get

$$m_*(\mathcal{O} \setminus E) \leq \epsilon.$$

and thus E is measurable. □

Property 3 *A countable union of measurable sets is still measurable.*

Proof. Let us assume that $E = \bigcup_j E_j$ where E_j is measurable for each j. By measurability, for each j we may find an open set $\mathcal{O}_j$ with

$$E_j \subset \mathcal{O}_j$$

such that

$$m_*(\mathcal{O}_j \setminus E_j) \leq \frac{\epsilon}{2^j}.$$

Then the union $\bigcup_j \mathcal{O}_j$ is open and contains E. Then

$$\mathcal{O} \setminus E \subset \bigcup_j (\mathcal{O}_j \setminus E_j).$$

Thus by monotonicity and sub-additivity of the exterior measure, we get

$$\begin{aligned} m_*(\mathcal{O} \setminus E) &\leq \sum_j m_*(\mathcal{O}_j \setminus E_j) \\ &\leq \epsilon. \end{aligned}$$

as desired. □

Property 4 *Closed sets are measurable*

Proof. First, note it suffices to prove that compact sets are measurable. As a matter of fact, any closed set F can be written as a union of compact sets. I.e., one can write

$$F = \bigcap_k F \cap B_k$$

where B_k are balls centered at the origin of radius k. Suppose then that F is compact and thus has finite exterior measure. Let $\epsilon > 0$, then by Observation 3 one can find an open set $\mathcal{O}$ containing F such that

$$m_*(\mathcal{O}) \leq m_*(F) + \epsilon.$$

As F is closed, $\mathcal{O} \setminus F$ is open by definition. By a theorem, we can write $\mathcal{O}$ as the countable union of almost disjoint cubes. That is,

$$\mathcal{O} \setminus F = \bigcup_j Q_j.$$

For any fixed $N \in \mathbb{N}$, the finite union

$$\bigcup_{j=1}^{N} Q_j$$

is compact. Thus $d(K, F) > 0$. By observations 4,5,6 we get that

$$\begin{aligned} m_*(\mathcal{O}) &\geq m_*(F) + m_*(K) \\ &= m_*(F) + \sum_j m_*(Q_j). \end{aligned}$$

Hence

$$\begin{aligned} \sum_j m_*(Q_j) &\leq m_*(\mathcal{O}) - m_*(F) \\ &\leq \epsilon. \end{aligned}$$

which holds as N tends to ∞. By sub-additivity we get

$$\begin{aligned} m_*(\mathcal{O} - F) &\leq \sum_j m_*(Q_j) \\ &\leq \epsilon. \end{aligned}$$

as we needed. □

We give a Lemma relating closed and compact set.

Lemma 5 *If F is closed and K is compact and the sets are disjoint, then*

$$d(F, K) > 0.$$

Proof. Since F is a closed set, for each $x \in K$, there exists some $\delta_x > 0$ such that

$$d(x, F) > 3\delta_x.$$

Note that by our choice of δ_x, we have that

$$K \subset \bigcup_{x \in K} B_{2\delta_x}(x).$$

By the compactness of K, there exists some finite subset $A \subset K$ such that

$$K \subset \bigcup_{x \in A} B_{2\delta_x}(x).$$

Take δ to be the minimum of the δ_x for $x \in A$, then

$$d(F, K) \geq \delta > 0.$$

Then if $x \in K$ and if $y \in F$, then for some $x_0 \in A$ we get

$$|x_0 - x| \leq 2\delta_0.$$

and by construction we have

$$\begin{aligned} |y - x| &\geq |y - x_0| - |x_0 - x| \\ &\geq 3\delta_0 - 2\delta_0 \\ &\geq \delta. \end{aligned}$$

as needed. □

Property 5 *The complement of a measurable set it measurable*

Proof. If $E \subset \mathbb{R}^d$ is measurable, then for every $n \in \mathbb{N}$ we may choose an open set $\mathcal{O}_n$ with $E \subset \mathcal{O}_n$ and

$$m_*(\mathcal{O}_n \setminus E) \leq \frac{1}{n}.$$

Since $\mathcal{O}_n$ is open for each n, its complement $\mathcal{O}_n^c$ is closed thus measurable. Thus $U = \bigcup_n \mathcal{O}_n^c$ is measurable as well. Note just by construction we have that

$$U \subset E^c.$$

Also note

$$E^c \setminus U \subset \mathcal{O}_n \setminus E.$$

Then by monotonicity, we get that

$$m_*(E^c \setminus U) \leq \frac{1}{n}$$

This gives us a measure zero set $E^c \setminus U$ which is measurable and equal to the union of U and $E^c \setminus U$ thus E^c is measurable. □

Property 6 *A countable intersection of measurable sets is measurable*

Proof. This follows from Properties 3,5 and by set theory since we know that

$$\left(\bigcap_j E_j\right) = \left(\bigcup_j E_j\right)^c.$$

□

Inc conlusion, measurable sets are closed under the basicv operations of set theory. Furhtermore, we have shown closures over *countable* unions and intersections, not just finite. Note measurable sets do not behave nicely with respect to *uncountable* unions and intersections. This leads us to our next big result.

Theorem 6 *If $E_1, E_2, \ldots$ are disjoint measurable sets, then we have the following equality*

$$m(E) = \sum_{j=1}^{\infty} m(E_j).$$

Proof. We may first assume that the E_j are bounded for each j. By Property 5, then we have that for each j, E_j^c is also measurable. Next we fix some $\epsilon > 0$ then choose closed subsets $F_j \subset E_j$ such that

$$m(E_j \setminus F_j) < \frac{\epsilon}{2^j}.$$

As the F_j are compact, then for any finite subset $A \subset \mathbb{N}$, we now that $\{F_j\}_{j\in A}$ forms a compact collection of sets. Furthermore, note the F_j are disjoint since they are inside of the E_j which were assumed to be disjoint. Thus we have by addivitiy that

$$m(\bigcup_{j\in A} F_j) = \sum_{j\in A} m(F_j).$$

And since $\bigcup_{j\in A} F_j \subset E$, then by monotonocity we get that

$$\begin{aligned} m(E) &\geq \sum_{j\in A} m(F_j) \\ &\geq \sum_{j\in A} m(E_j) + \epsilon. \end{aligned}$$

The reverse inequality also holds by countable sub-additivity and this case is complete.

Next we show the general case in which the E_j are not-bounded. We select a sequence of cubes $\{Q_j\}_{j\in\mathbb{N}}$ which tends to all of $\mathbb{R}^d$. That is, $Q_j \subset Q_{j+1}$ for all j and that the union of these cubes is all of $\mathbb{R}^d$. So we define a new sequence. Take

$$S_1 := Q_1, S_j := Q_j \setminus Q_{j-1}.$$

The latter for $j \geq 2$.. These S_j are clearly measurable as they are complements of measurable sets. We can then define measurable unions of sets via

$$E_{k,j} = E_k \cap S_j.$$

Then we have that

$$\bigcup_{k,j} E_{k,j},$$

which is a disjoint and bounded union and note that $E_j = \bigcup_{k,j} E_{k,j}$. Using these facts together with what has been proven already, we obtain

$$\begin{aligned} m(E) &= \sum_{k,j} E_{k,j} \\ &= \sum_k \sum_j m(E_{k,j}) \\ &= \sum_k m(E_k). \end{aligned}$$

as desired. □

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§ 3.2 MCT for sets

Before jumping to the notion of Monotone converging measurable functions we will discuss what it means for monotone converging sets.

If $E_1, E_2, ...$ are measurable increasing sets in the senses that $E_k \subset E_{k+1}$ and if $E = \bigcup_k E_k$, we write $E_k \nearrow E$.

Similarly, if the sets $E_1, E_2, ...$ are decreasing in the senses that $E_k \supset E_{k+1}$ and if $E = \bigcap_k E_k$ we then write $E_k \searrow E$. Moreover, the latter condition requires at least one of the E_k, say E_1 has finite measure. That is, $m(E_1) < \infty$. This leads use to a major corollary.

Corollary 7 *If $E_1, E_2, ...$ are measurable subsets of $\mathbb{R}^d$. Then*

- *If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_k)$.*
- *If $E_k \searrow E$ and if $m(E_k) < +\infty$ for some k, then $m(E) = \lim_{N\to\infty} m(E_N)$.*

Proof. For the proof of (a), we define measurable sets $G_1 = E_1$ for $k = 1$, then for every $k \geq 2$ define

$$G_k := E_k \setminus E_{k-1}$$

By construction, the G_k are measurable, and disjoint. Moreover, note that $E = \bigcup_k G_k$. Then we have

$$\begin{aligned}
m(E) &= m(\bigcup_k G_k) \\
&= \sum_{k\in\mathbb{N}} m(G_k) \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k) \\
&= \lim_{N\to\infty} m(\bigcup_{k=1}^{N} G_k) \\
&= \lim_{N\to\infty} m(\bigcup_{k=1}^{N} E_k) \\
&= \lim_{N\to\infty} m(E_N).
\end{aligned}$$

as desired.

For (b), note since the E_k are decreasing, then

$$E_1 \setminus E_k \subset E_1 \setminus E_{k+1}.$$

Is an increasing sequence. First note from basic set theory, one has

$$\begin{aligned}
\bigcup E_1 \setminus E_k &= \bigcup E_1 \cap E_k^c \\
&= E_1 \cap \bigcup E_k^c \\
&= E_1 \cap (\bigcap E_k)^c \\
&= E_1 \setminus \bigcap_k E_k
\end{aligned}$$

Then we can compute

$$\begin{aligned}
m(E_1 \setminus \bigcap E_k) &= m(E_1) - m(\bigcap E_k) \\
&= m(\bigcup E_1 \setminus E_k) \\
&= \lim_{n\to\infty} m(E_1 \setminus E_k) \\
&= m(E_1) - \lim_{n\to\infty} m(E_n)
\end{aligned}$$

and since $m(E_1) < +\infty$, we can subtract it from both sides to obtain the desired result. □

The following theorem will play a central roll in approximations of measurable functions.

Theorem 8 *Suppose $E \subset \mathbb{R}^d$ is measurable. Then, for every $\epsilon > 0$,*

- *There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$, and*

$$m(\mathcal{O} \setminus E) \leq \epsilon.$$

- *There exists a closed set F with $F \subset E$ and*

$$m(E \setminus F) \leq \epsilon.$$

- *If $m(E) < +\infty$, then there exists a compact set K with $K \subset E$ and*

$$m(E \setminus K) \leq \epsilon.$$

- *If $m(E) < +\infty$, then there exists a finite union of $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that the symmetric difference of E and F is less than or equal to ϵ.*

Proof. For (a), this just follows from definitions of measurability. For (b), note that since E is measurable, then so is E^c. So there exists an open set containing E^c within ϵ. So there exists some open set $\mathcal{O}$ with $E^c \subset \mathcal{O}$ such that

$$m(\mathcal{O} \setminus E^c) \leq \epsilon.$$

Since $\mathcal{O}$ is open, its complement $\mathcal{O}^c$ is closed and contained inside of E. Moreover, one has that $E \setminus \mathcal{O}^c = \mathcal{O} \setminus E^c$ and thus

$$m(E \setminus \mathcal{O}^c) \leq \epsilon.$$

as needed. For (c), we first pick a closed set F such that

$$m(E \setminus F) \leq \epsilon.$$

For each $n \in \mathbb{N}$, let B_n denote the ball centered at the origin of radius n. Define compact sets via

$$K_n := F \cap B_n.$$

Then $E \setminus K_n$ is a measurable sequence of decreasing sets to $E \setminus F$. Since $m(E) < +\infty$, we conclude for very large $n \in \mathbb{N}$ that

$$m(E \setminus K_n) \leq \epsilon.$$

For the last part, we choose a family of closed cubes $\{Q_j\}_{j \in \mathbb{N}}$ so that

$$E \subset \bigcup_{j \in \mathbb{N}} Q_j,$$

and that

$$\sum_{j \in \mathbb{N}} |Q_j| \leq m(E) + \frac{\epsilon}{2}.$$

Since $m(E) < +\infty$, the series converges and there exists an $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} |Q_j| \leq \frac{\epsilon}{2}.$$

If there exists some finite subset $A \subset \mathbb{N}$ where $F = \bigcup_{j \in A} Q_j$, then one can compute

$$\begin{aligned} m(E \triangle F) &= m(E \setminus F) + m(F \setminus E) \\ &\leq m\Big(\bigcup_{j=N+1}^{\infty} Q_j\Big) + m\Big(\bigcup_{j=1}^{\infty} Q_j \setminus E\Big) \\ &\leq \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \\ &\leq \epsilon. \end{aligned}$$

as $\epsilon > 0$ was arbitrarily chosen, we are done. □

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§ 3.3 Invariance Properties

Often times we would like to know which properties are invariant under the measure function, m. One key property that is preserved is this notion of translation invariance. That is, one would like to know the measure of the set $\{x + h : x \in E, h \in \mathbb{R}^d\}$, in fact the measure is preserved in the following sense:

$$m(E_h) = m(E),$$

where $E_h := \{x + h : x \in E, h \in \mathbb{R}^d\}$. Suppose we are given some $\delta > 0$, then the set

$$\delta E := \{\delta x : x \in E\},$$

has (Lebesgue) measure

$$m(\delta E) = \delta^d m(E).$$

Furthermore, note that the measure function is reflection invariant as well. That is, if

$$-E = \{-x : x \in E\},$$

then

$$m(-E) = m(E).$$

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§ 4 $\sigma-$algebras & Borel sets

Definition: A $\sigma-$*algebras* of sets is a collection of subsets closed under countable unions, intersections, and complements. The collection of all $\sigma-$algebras is of course a $\sigma-$algebra. We will consider a particular collection of $\sigma-$algebras called the *Borel* $\sigma-$*algebra in* $\mathbb{R}^d$. This Borel $\sigma-$algebra will be the "smallest" $\sigma-$algebra containing all of the open subsets of $\mathbb{R}^d$.

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§ 4.1 Borel Sets

From the view point of the Borel sets, the Lebesgue sets arise from the *completion* of the $\sigma-$algebra of the Borel sets, that is, by adjoining all Borel sets of measure zero.

Starting with open and closed sets, which are the most simplest Borel sets, one could list the Borel sets in order of their complexity. Next in order would be countable intersections of open sets, called G_δ sets and their respective complemenets, the countable union of closed sets, called the F_σ sets. This gives rise to our next Corollary.

Corollary 9 *A subset $E \subseteq \mathbb{R}^d$ is measurable*

- *if and only if E differs from a G_δ by a set of measure zero*
- *if and only if E differs from a F_σ by a set of measure zero*

Proof. Clearly E is measurable whenever it satisfies the first or second. This is because the F_σ, G_δ, and sets of measure zero are all measurable. On the other hand, if E is measurable, then for each $n \in \mathbb{N}$, we may select an open $\mathcal{O}_n$ of E such that

$$m(\mathcal{O}_n \setminus .E) \leq \frac{1}{n}.$$

Then $S = \bigcup_{n\in\mathbb{N}} \mathcal{O}_n$ is a G_δ set containing E such that

$$S \setminus E \subset \mathcal{O}_n \setminus E$$

for every $n \in \mathbb{N}$ thus by monotonicity, we get

$$m(S \setminus E) \leq \frac{1}{n},$$

has zero exterior measure and is therefore measurable. For the second implication, we invoke **Theorem 8(ii)** with $\epsilon = \frac{1}{n}$. □

§ **5 Construction of non-measurable set.**

It turns out that not all subsets of $\mathbb{R}^d$ are in fact measurable. In this section, we give the construction of a non-measurable set $\mathcal{N}$. We will consider the case when $d = 1$, and consider the following subset of $\mathbb{R}$. We place an equivalence relation $\sim$ on [0,1] by putting $x \sim y$ whenever $x - y \in \mathbb{Q}$. We quickly get the three properties of an equivalence relation, that is $\sim$ is reflexive, symmetric, and transitive. Two equivalence classes are either the same or disjoint and we can write [0,1] as the union over all equivalence classes we write as

$$[0,1] = \bigcup_\alpha \mathcal{E}_\alpha.$$

Next, we construct our set $\mathcal{N}$ by selecting one element $x_\alpha \in \mathcal{E}_\alpha$ from each equivalence class. That is, we write

$$\mathcal{N} = \{x_\alpha\}.$$

This leads us to our big theorem of this section:

Theorem 10 *The set $\mathcal{N}$ is non-measurable.*

Proof. We prove this theorem by contradiction. We assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_k$ be an enumeration of $\mathbb{Q} \cap [-1, 1]$. Consider the translates

$$\mathcal{N}_k = \mathcal{N} + r_k.$$

I claim the $\mathcal{N}_k$ are disjoint and that

$$[0, 1] \subset \bigcup_{k \in \mathbb{N}} \mathcal{N} + r_k \subset [-1, 2].$$

To see why the sets are disjoint, consider the intersection

$$\mathcal{N}_k \cap \mathcal{N}_{k'}$$

being non-empty, then there exists rationals $r_k, r_{k'}$ that are distinct and α, β with

$$x_\alpha + r_k = x_\beta + r_{k'}.$$

Then we have

$$x_\alpha - x_\beta = r_{k'} - r_k.$$

And thus $\alpha \neq \beta$ and $x_\alpha - x_\beta \in \mathbb{Q}$ hence $x_\alpha \sim x_\beta$ which contradicts $\mathcal{N}$ containing only one representative from each equivalence class. The inclusion

$$\bigcup_{k \in \mathbb{N}} \mathcal{N}_k$$

is immediate by construction. To see why $[0, 1] \subset \bigcup_{k \in \mathbb{N}} \mathcal{N}_k$, let $x \in [0, 1]$. Then $x \sim x_\alpha$ for some α thus

$$x - x_\alpha = r_k$$

for some $k \in \mathbb{N}$. Thus $x \in \mathcal{N}_k$ and we get the first inclusion. If $\mathcal{N}$ is measurable, then so is $\mathcal{N}_k$ for each k and by monotonicity, we have that

$$1 \leq \sum_{k \in \mathbb{N}} m(\mathcal{N}_k) \leq 3.$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we have $m(\mathcal{N}) = m(\mathcal{N}_k)$ for every k. We conclude that

$$1 \leq \sum_{k \in \mathbb{N}} m(\mathcal{N}) \leq 3.$$

This contradicts our set $\mathcal{N}$ being measurable since it can neither have measure zero or infinite measure. □

§ 5.2 Axiom of Choice

The construction of the set $\mathcal{N}$ is possible based on the following general proposition.

Proposition 11 Suppose E is a set and $\{E_\alpha\}_{\alpha\in A}$ is a collection of non-empty subsets of E. (Here, the indexing set A is not assumed to be countable). Then there exists a "choice" function

$$\alpha \to x_\alpha,$$

such that $x_\alpha \in E_\alpha$ for every $\alpha \in A$.

Definition: In this general form, this assertion is known as the *axiom of choice*. An initial use of this mere fact was used to prove the *well-ordering* principle. We make this a bit more rigorous as follows.

Definition: A set E is *linearly ordered* if there exists a binary relation $\leq$ on E such that

(a) $x \leq x$; $\forall x \in E$

(b) If $x, y \in E$ are distinct, then either $x \leq y$ or $y \leq x$ but not both.

(c) If $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition: We say that a set E can be *well-ordered* if it can be linearly ordered in such a way that *every* non-empty subset $A \subset E$ has a smallest element in that ordering. That is an element $x_0 \in A$ such that $x_0 \leq x$ for every other $x \in A$. In general, any set E can be well-ordered. It is in fact nearly obvious that the well-ordering principle implies the axiom of choice: If we well order E, then we can choose x_α to be the smallest element of E_α. Conversely, the axiom of choice implies the well-ordering principle.

§ **6 Measurable functions** For $x \in E$, we begin with the *characteristic function* of a given set $E \subset \mathbb{R}^d$. That is, define

$$\chi_E(x) = \begin{cases} 0 & ; x \notin E \\ 1 & ; x \in E. \end{cases}$$

We then lean over toward the building blocks of the Riemann integral. For the Riemann integral, these will be *step functions*. A **step function** is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}(x)$$

Where each R_j is a rectangle. A **simple function** is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{R_k},$$

where each R_k is measurable of finite measure, and the a_k are constants.

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§ 6.1 Properties of finite- valued

We begin by considering those subsets of $\mathbb{R}^d$ that take on real-values. That is, for any $x \in E \subset \mathbb{R}^d$, we have

$$-\infty \leq f(x) \leq +\infty,$$

in the case of finite-valued functions f, we say that f is finite-valued if the inequality is strict.

$$-\infty < f(x) < +\infty.$$

Definition: A function defined on some measurable subset $E \subset \mathbb{R}^d$, is said to be *measurable* if for every $a \in \mathbb{R}$, the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable. To simplify this notion we often just write

$$\{x \in E : f(x) < a\} = \{f < a\}.$$

Note first off that there are many equivalent definitions of a measurable function for us now given complemenets. For example, we may require the inverse image of closed intervals need be measurable. In fact, to prove f is measurable if and only if $\{x \in E : f(x) \leq a\} = \{f \leq a\}$ is measurable for every $a \in \mathbb{R}$, note in one direction one has

$$\{f \leq a\} = \bigcap_{k=1}^{\infty} \{f < a + \frac{1}{k}\}.$$

Recall a countable union of measurable sets need be measurable thus for the other direction, observe that

$$\{f < a\} = \bigcup_{k=1}^{\infty} \{f \leq a - \frac{1}{k}\}.$$

Similarly, f is measurable if and only if $\{f \leq a\}$ or $\{f > a\}$ is measurable for every $a \in \mathbb{R}$. This is immediate from the fact that

$$\{f \geq a\} = \{f < a\}^c$$

and in the second case we have that

$$\{f \leq a\} = \{f > a\}^c.$$

Consequently, $-f$ is measurable whenever f is. In particular, one can show if f is finite-valued (that is, $-\infty < f(x) < +\infty$.), then f is measurable if and only if the set

$$\{a < f < b\}$$

is measurable for every $a, b \in \mathbb{R}$. This leads us to our next few properties:

> **Property 1** *The finite-valued function $f(x)$ is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$, and if and only if $f^{-1}(F)$ is measurable for every closed set F.*

> **Property 2** *If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued and Φ is continuous, then $\Phi \circ f$ is continuous.*

By continuity of Φ, we have that

$$\Phi^{-1}((-\infty, a))$$

is an open set, call it $\mathcal{O}$. Hence

$$(\Phi \circ f)^{-1} = f^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O}).$$

is measurable.

Property 3 *Suppose* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of measurable functions, then*

$$\sup_n f_n(x),\ \inf_n f_n(x),\ \lim_{n\to\infty} \sup, f_n(x),\ \lim_{n\to\infty} \inf f_n(x)$$

are measurable.

Property 4 *If* $\{f_n\}_{n=1}^{\infty}$ *is a collection of measurable functions and*

$$\lim_{n\to\infty} f_n(x) = f(x),$$

Then $f(x)$ *is measurable.*

Property 5 *If* f *and* g *are measurable, then*

(a) The integer powers f^k *for* $k \geq 1$ *are measurable.*

(b) $f+g$, fg *are measurable if both* f, g *are both finite-valued.*

Proof. To see (a), note if k is odd, then $\{f^k > a\} = \{f > a^{\frac{1}{k}}\}$, and if k is even and $a \geq 0$, then

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\}.$$

For (b), note that to see why $f+g$ is measurable, we can write

$$\{f+g > a\} = \bigcup_{r\in\mathbb{Q}} \{f > a-r\} \cap \{g > r\}.$$

To see why the product is measurable, note that

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2].$$

□

Definition: we shall say two functions f, g defined on a set E are equal *almost everywhere* if the set

$$\{x \in E : f(x) \neq g(x)\}$$

has measure zero.

Note that if f, g are defined almost everywhere on measurable subset of $\mathbb{R}^d$, then the functions $f+g$, fg can only be defined on the intersection of the domains of f and g. We summarize this fact with the following property:

Property 6 *Suppose* f *is measurable and* $f(x) = g(x)$ *a.e. on* E*. Then* g *is measurable.*

Proof. To show this, we must show for any measurable set $E \subseteq \mathbb{R}^d$, $g(E) \in \mathcal{M}$. That, is $g^{-1}(E)$ is measurable. Note then that

$$\begin{aligned} g^{-1}(E) &= \{x : g(x) \in E\} \\ &= \{x : g(x) \in E, f(x) = g(x)\} \cup \{x : g(x) \in E, f(x) \neq g(x)\} \end{aligned}$$

Since $f = g$ a.e., this tells us the latter of the two sets is a null set. Moreover, if A is a measurable set in the range of g, then we can write

$$\begin{aligned} A &= \{x : g(x) \in E, f(x) = g(x)\} \\ &= \{x : f(x) \in E, f(x) = g(x)\} \\ &= \{x : f(x) \in E\} \cap \{x : f(x) = g(x)\}. \end{aligned}$$

Here the first of the two sets is measurable as f is measurable, and the second set is the complement of a measurable set and is thus measurable. □

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§ 7 Approximation via simple and step functions.

A lot of the theorems in this section are of the same nature. We begin by approximating point-wise, non-negative measurable functions by simple functions. This leads us to our first big theorem of this section.

Theorem 12 *Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that converges pointwise to f, namely,*

$$\varphi_k \leq \varphi_{k+1} \text{and} \lim_{k\to\infty} \varphi_k(x) = f(x)$$

for all x.

Proof. We first begin with a truncation. For $N \geq 1$, let Q_N denote the cube centered at the origin of side length N. Then define

$$F_N(x) = \begin{cases} f(x) & ; x \in Q_N, f(x) \leq N \\ N & ; x \in E, f(x) > N \\ 0 & \text{otherwise} \end{cases}.$$

Then it is clear that

$$F_N \to f$$

as $N \to \infty$. Next, we partition the range of $F_N(x)$. Namely, $[0, N]$, as follows. For fixed $M, N \geq 1$, we define

$$E_{l,M} := \{x \in Q_N : \frac{l}{m} < F_N(x) \leq \frac{l+1}{M}\},$$

for $l \in [0, MN)$. Then we can define

$$F_{N,M}(x) = \sum_l \frac{l}{M} \chi_{E_{l,M}}(x).$$

Where each $F_{N,M}$ is a simple function satisfying

$$0 \leq F_N(x) - F_{N,M}(x) \leq \frac{1}{M},$$

for every x. If we choose $N = M = 2^k$ with $k \geq 1$, and let $\varphi_k = F_{2^k,2^k}$, then we see that

$$0 \leq F_M(x) - \varphi_k(x) \leq \frac{1}{2^k}$$

for every x, and $\{\varphi_k\}$ is increasing and satisfies the desired properties. □

Note that the result holds for non-negative functions that are extended valued, if the limit ∞ is allowed. We now drop the assumption that f be non-negative.

Theorem 13 *Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that satisfy*

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)|$$

and

$$\lim_{k=1}^{\infty} \varphi_k(x) = f(x),$$

for all x.

Proof. We use the decomposition of f into f^+ and f^-. That is, we can write

$$f(x) = f^+(x) - f^-(x)$$

where

$$f^+(x) := \max(f(x), 0), f^-(x) := \max(-f(x), 0).$$

Since both functions are non-negative, the previous result gives us two increasing sequences of non-negative simple functions $\{\varphi_k^{(1)}\}_{k=1}^{\infty}$ and $\{\varphi_k^{(2)}\}_{k=1}^{\infty}$ converging point-wise to f^+ and f^- respectively. Then if we take

$$\varphi_k(x) := \varphi_k^{(1)}(x) - \varphi_k^{(2)}(x)$$

we see that $\varphi_k \to f$ for every x. Finally, the sequence $\{|\varphi_k|\}$ is increasing by definitions of f^+, f^- and properties of $\varphi_k^{(1)}$, $\varphi_k^{(2)}$ imply that

$$|\varphi_k(x)| = \varphi_k^{(1)}(x) + \varphi_k^{(2)}(x)$$

The next step, is to approximate by step functions. Here, in general, the convergence may hold only a.e. □

Theorem 14 *Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges point-wise to $f(x)$ for almost every x.*

Proof. By the previous theorem, there are simple functions $\{\varphi_k\}_{k=1}^{\infty}$ such that

$$\lim_{k=1}^{\infty} \varphi_k(x) = f(x)$$

for every x. To approximate each φ_k by a step function, recall Theorem 8 (iv) which states if E is measurable of finite measure, then for any $\epsilon > 0$ there exists a finite subset $A \subset \mathbb{N}$ such that

$$m(E \bigtriangleup \bigcup_{k \in A} Q_k) \leq \epsilon.$$

By considering the grid formed by extending the sides of the cubes, we see there exists almost disjoint rectnalges $\overline{R}_1, ..., \overline{R}_M$ such that

$$\bigcup_{k\in A} Q_k = \bigcup_{j\in B} \overline{R}_j$$

By taking closed rectangles R_j contained in $\overline{R}_j$ and slightly smaller in size, we find a collection of disjoint closed rectangles that satisfy

$$m(E \triangle \bigcup_{j\in B} R_j) \leq 2\epsilon.$$

So then by definition of a simple function that for each k, there exists a step function ψ_k, and a measurable set F_k such that $m(F_k) < \frac{1}{2^k}$ and that

$$\varphi_k(x) = \psi_k(x)$$

for every $x \notin F_k$. If we define

$$F : \bigcap_{l=1}^{\infty} \bigcup_{k>l}^{\infty} F_k,$$

then $m(F) = 0$ since

$$m(\bigcup_{k>l}^{\infty} F_k) \leq \sum_{k>l} m(F_k) \leq \frac{1}{2^l}.$$

For $x \notin F$, there exists some $k_0 \in \mathbb{N}$ such that

$$x \in \bigcap_{k>k_0} F_k^c,$$

thus for every $k > k_0$, one has

$$\begin{aligned} |f(x) - \psi_k(x)| &\leq |f(x) - \varphi_k(x)| + |\varphi_k(x) - \psi_k(x)| \\ &= |f(x) - \varphi_k(x)| \end{aligned}$$

and since $\varphi_k \to f$ as $n \to \infty$, we conclude that

$$\lim_{k\to\infty} \psi_k(x) = f(x)$$

for every $x \notin F$ as desired. □

§ **8 Littlewood's Three principles.**

Although the notion of measurable sets and functions represents new tools, we should not overlook their relation to the older concepts they replaced. Littlewood aptly summarized these connections in the form of three principles that provide a useful intuitive guide in the initial study of the theory.

(a) Every set is nearly a finite union of intervals.

(b) Every function is nearly continuous.

(c) Every convergent sequence is nearly uniformly convergent.

The sets and functions referred to above are of course assumed to be measurable. The catch is in the word "nearly". A precise version of the first principle appears in part (iv) of Theorem 8. An important formulation of the third principle appears in the following result.

Theorem 15 *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$ and assume $f_k \to f$ a.e. on E. Given $\epsilon > 0$ we can always find a closed set $A_\epsilon \subset E$ such that*

$$m(E \setminus A_\epsilon) \leq \epsilon,$$

and $f_k \to f$ uniformly on A_ϵ.

Proof. We may assume, without any loss of generality, that $f_k \to f$ for every $x \in E$. For each pair of non-negative integers n, k let

$$E_k^n := \{x \in E : |f_j(x) - f(x)| < \frac{1}{n}\}.$$

Now, fix $n \in \mathbb{N}$ and note that the E_k^n are increasing in set containment. That is,

$$E_k^n \subset E_{k+1}^n$$

for every k. Furthermore, note that $E_k^n \nearrow E$. By MCT for measurable sets, there exists some $k_n \in \mathbb{N}$ such that

$$m(E \setminus E_{k_n}^n) \leq \epsilon.$$

By construction then we have that

$$|f_j(x) - f(x)| < \frac{1}{n}$$

whenever $j > k_n$ and $x \in E_{k_n}^n$. Next, we select our $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2},$$

and let

$$\tilde{A}_\epsilon = \bigcap_{n \geq \mathbb{N}} E_{k_n}^n.$$

Then note

$$\begin{aligned} m(Em \setminus \tilde{A}_\epsilon) &\leq \sum_{n=N}^{\infty} m(E \setminus E_{k_n}^n) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Next, if $\delta > 0$, we can choose $n \geq N$ such that $\frac{1}{n} < \delta$ and note that $x \in \tilde{A}_\epsilon$ implies $x \in E_{k_n}^n$. So then whenever $j > k_n$, we have

$$|f_j(x) - f(x)| < \delta.$$

Thus $f_k \to f$ uniformly on A_ϵ. Finally, by Theorem 8, we can choose a closed subset $A_\epsilon \subset \tilde{A}_\epsilon$ with

$$m(\tilde{A}_\epsilon \setminus A_\epsilon) < \frac{\epsilon}{2}$$

This results in

$$m(E \setminus \tilde{A}_\epsilon) < \epsilon.$$

as desired. □

The next result attests to the second of three principles.

Theorem 16 (Lusin) *Suppose f is measurable and finite valued on E with E having finite measure. Then for every $\epsilon > 0$, there exists a closed set F_ϵ with*

$$F_\epsilon \subset E$$

and

$$m(E \setminus F_\epsilon) \leq \epsilon,$$

and such that $f|_{F_\epsilon}$ is continuous.

Proof. Let f_n be a sequence of step functions such that

$$f_n \to f$$

a.e. x. Then we can find sets E_n such that $m(E_n) < \frac{1}{2^n}$ and f_n is continuous outside of E_n. By Egorov's Theorem, we can find a closed set $A_{\frac{\epsilon}{3}} \subset E$ on which $f_n \to f$ uniformly and

$$m(E \setminus A_{\frac{\epsilon}{3}}) \leq \frac{\epsilon}{3}.$$

We then consider the the set

$$F' = A_{\frac{\epsilon}{3}} \setminus \bigcup_{n \geq N} E_n$$

for N so large that

$$\sum_{n \geq N} \frac{1}{2^n} < \frac{\epsilon}{3}.$$

Now for all $n \geq N$, the function f_n is continuous on F', thus f is also continuous on F'. □

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