Informal Notes: Several Complex Variables

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Notes pulled from:
Complex Geometry by Daniel Huybrechts;
Functions of One Complex Variable I by John B. Conway;
Algebraic Curves and Riemann surfaces by Rick Miranda
Principles of Algebraic Geometry by Phillip Griffiths and Joseph Harris

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Abstract

In these notes we give a brief overview at the theory of functions of several complex variables. The only prior knowledge to have to make progress in these notes is basic point set topology, introductory Complex Analysis (analytic functions, Cauchy-Riemann equations, Liouvilles Theorem.), Abstract Algebra, and some PDEs. Our goal is to approach the theory first from that of a single complex variable then generalize to higher dimensions whilst trying to not loose track of any complex or algebraic structure.

Acknowledgements

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§1. Holomorphic Functions

The theory of functions of several complex variables is a rich and elegant field; it pulls from various fields such as but not limited to, Topology, Abstract Algebra, and Partial differential equations. One of the most crucial concepts across the board for complex variables is the notion of a *holomorphic* map. Before moving into the land of several complex variables, we must first recall what it means for a function defined over $\mathbb{C}$ to be holomorphic. For the remainder of these notes, $U \in \tau_{\mathbb{C}}$ when defined. That is, U is open in $\mathbb{C}$. Also we will consider the reader knows that $\mathbb{C}$ is a field and under the usual product operations we have that $i = \sqrt{-1}$ which can be left as an exercise for the reader. Additionally, note for $z \in \mathbb{C}$ one can write

$$z = x + iy,$$

where $x, y \in \mathbb{R}$. Then one can define the *complex conjugate* of z by

$$\overline{z} = x - iy.$$

Exercise 0. Show in $\mathbb{C}$, $i = \sqrt{-1}$. (Hint: $(a, b)(c, d) = (ac - bd, ad + bc)$).

Let $U \subset \mathbb{C}$. We say that a map

$$f : U \to \mathbb{C}$$

is *holomorphic* if for each $z \in U$,

$$\frac{\partial f}{\partial \overline{z}} = 0$$

Equivalently, one could say

$$\lim_{h \to \infty} \frac{f(z+h) - f(z)}{h}$$

exists. Note that here $0 \neq h \in \mathbb{C}$ (Here, h could tend to zero from an uncountable number of directions. This is one of the big reasons why being continuous over $\mathbb{C}$ is much stronger than being continuous over $\mathbb{R}$. Over the reals one only need to ensure the left and right sided limits exists and are identical, over the complex plane however, a point can be approached from any of the angles within 360.) and $z + h \in U$.

We say f is *analytic* on U if for each $z_0 \in U$ there exists an $\epsilon > 0$ such that $f(z)$ converges on $B_\epsilon(z_0)$. Moreover, f is analytic if it can be written as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$$

for every $z \in B_\epsilon(z_0)$.

Theorem 0. Let $f \in C^\infty(U)$. Then f is holomorphic if and only if f is analytic.

Proof. First let us suppose f is holomorphic on U. That is, for every $z \in U$ we have that

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This by defintion is a limit thus we can find and $\epsilon > 0$ such that for $z_0 \in U$ and $z \in B_\epsilon(z_0)$, by the Cauchy integral formula we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(w)}{w-z}dw \\ &= \int \partial B_\epsilon(z_0)\frac{f(w)}{(w-z_0)-(z-z_0)}dw \\ &= \int \partial B_\epsilon(z_0)\frac{f(w)}{(w-z_0)-(1-\frac{z-z_0}{w-w_0})}dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n. \end{aligned}$$

Thus setting the coefficients equal to

$$a_n = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw,$$

we have that

$$f(x) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for $z \in B_\epsilon(z_0)$ where the sum converges uniformly and absolutely on any small disk.

On the other hand, suppose f has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

which converge for every $z \in U$ which in our case we can let be $B_\epsilon(z_0)$ Rest of Proof coming soon, being pulled from Griffiths & Harris □

There are many equivalent definitions of holomorphicity over $\mathbb{C}$, however one definition we appeal to most is that of the *Cauchy-Riemann* equations. Recall that $z \in \mathbb{C}$ can be written as

$$z = x + iy$$

where $x, y \in \mathbb{R}$. Then f can be regarded as a function $f(x, y)$ of two real variables. As a matter of fact one can write f as

$$f(x,y) = u(x,y) + iv(x,y)$$

where

$$u(x,y) := \mathrm{Re}(f), v(x,y) := \mathrm{Im}(f),$$

the real and imaginary parts respectively. Note that u, v are real-valued functions

$$u, v : \mathbb{R}\times\mathbb{R} \to \mathbb{R}.$$

Now one can show that f is holomorphic if and only if the Cauchy-Riemann equations are satisfied. That is, if and only if

$$u_x = v_y$$
$$u_y = -v_x.$$

I.e., the derivative of f need be $\mathbb{C}-$linear. This allows us to define differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \qquad (*)$$

and

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}). \qquad (**)$$

These are motivated by the properties

$$\frac{\partial}{\partial z}(z) = 1 = \frac{\partial}{\partial \overline{z}}(\overline{z}),$$

and

$$\frac{\partial}{\partial z}(\overline{z}) = 0 = \frac{\partial}{\partial \overline{z}}(z),$$

Then the C-R equations can be writtten as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

As the jump from real partial derivatives to complex partial derivatives is vast, we will spend a bit more time on this section. Consider a differentiable map

$$f : U \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2.$$

Then it only makes sense to mention the differential of f at some $z \in U$. Namely, its differential $df(z)$ at some $z \in U$ is the $\mathbb{R}-$linear map

$$df(z) : T_z\mathbb{R}^2 \to T_{f(z)}\mathbb{R}^2$$

between tangent spaces. It is crucial to note the dimension of the tangent space is the same as the dimension of our ambient space. Writing $z = x + iy, w = r + is$ for the two tangent spaces we enjoy a nice canonical bases given via

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle, \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial s} \rangle.$$

With respect to these basis, the differential $df(z)$ is given via the real Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

where $f = u + iv$. I.e., $u = r \circ f$, $v = s \circ f$.

Our goal is to now extend this to a $\mathbb{C}-$linear map, (Recall a map f is $\mathbb{C}-$ linear if

$$f(i) = if(1).)$$

which is given via

$$df(z)_{\mathbb{C}} : T_z\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z)}\mathbb{R}^2 \otimes \mathbb{C}.$$

We can now choose as basis, $(*)$ and $(**)$. With respect to this basis, $df(z)$ is given via

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \\ \frac{\partial \overline{f}}{\partial z} & \frac{\partial \overline{f}}{\partial \overline{z}} \end{pmatrix}.$$

Exercise 1. Show that for any function f defined one some open $U \subset \mathbb{C}$,

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\left(\frac{\partial f}{\partial z}\right)}.$$

Exercise 2. If $f = u + iv$ is holomorphic with

$$u, v : \mathbb{R}^2 \to \mathbb{R},$$

and

$$u, v \in C$$

then

$$\frac{\partial f}{\partial \overline{z}} = 0 = \frac{\partial \overline{f}}{\partial z}.$$

Using results from these two exercises we note that $df(z)$ has a new base given via the diagonal matrix

$$\begin{pmatrix} \frac{\partial f}{\partial z} & 0 \\ 0 & \frac{\partial \overline{f}}{\partial \overline{z}} \end{pmatrix}.$$

Holomorphicity of f is also equivalent to the *Cauchy Integral Formula.* More precisely, a function

$$f : U \to \mathbb{C}$$

is holomorphif if and only if f is continuously differentiable and for any

$$B_\epsilon(z_0) \subset U,$$

we have that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{(z - z_0)} dz.$$

In fact, this formula holds for any function

$$f : \overline{B_\epsilon(z_0)} \to \mathbb{C}$$

which is holomorphic on the interior. Namely, the Cauchy integral formula is used in proving the existence of a (convergent) power series expansion of any function satisfying the Cauchy-Riemann equations.

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We list a few crucial results from functions of a single variable which will be of use to us.

Maximum Principle Let $U \subset \mathbb{C}$ be open and connected. If

$$f : U \to \mathbb{C}$$

is holomorphic and non-constant, $|f|$ has no local maximum in U. Moreover, if U is bounded and f can be extended to a continuous function

$$\overline{f} : \overline{U} \to \mathbb{C},$$

then $|f|$ takes on its maximal values on ∂U.

Maximum Principle (Alternate) Let $U \subset \mathbb{C}$ is open and connected If

$$f : U \to \mathbb{C}$$

is holomorphic and there exists a point $z_0 \in U$ such that

$$|f(z_0)| \geq |f(z)|$$

for every $z \in U$, then f is constant on U.

Identity Theorem If

$$f, g : U \to \mathbb{C}$$

are holomorphic functions on an open and connected subset $U \subset \mathbb{C}$ such that

$$f(z) = g(z)$$

for each $z \in V \subset U$ some non-empty open subset, then $f = g$.

Riemann extension Theorem Let

$$f : B_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$$

bee a bounded holomorphic function. Then, f can be extended to a holomorphic function

$$f : B_\epsilon(0) \to \mathbb{C}.$$

Note this only holds for the case $n \geq 2$.

Riemann Mapping Theorem Let $U \subsetneq \mathbb{C}$ be simply connected. Then $U \cong B_1(0)$. That is, there exists a bijective holomorphic map

$$f : U \to B_1(0)$$

such that f^{-1} is also holomorphic.

In other words, U is conformally equivalent to the open unit disk if U is not all of $\mathbb{C}$ (Tao, 2018).

Note this only holds for the case $n = 1$. To see why it fails for higher dimensions refer to this exercise

Liouville If

$$f : \mathbb{C} \to \mathbb{C}$$

is bounded, then f is constant. I.e., Bounded entire functions need be constant. Here entire is used to denote holomorphicity on all of $\mathbb{C}$.

Note if we swap $\mathbb{C}$ for $\mathbb{R}$ this does not hold. To see this, consider

$$f : \mathbb{R} \to \mathbb{R}$$

defined via

$$x \mapsto \sin x$$

which is clearly bounded, real-analytic but not constant (over $\mathbb{R}$.

Residue Theorem Let

$$f : B_\epsilon(0) \setminus \{0\} \to \mathbb{C}$$

be a holomorphic map. Then f can be extended via a Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where the coefficient a_{-1} is given by the residue formula

$$a_{-1} = \frac{1}{2\pi i} \int_{|z|=\frac{\epsilon}{2}} f(z) dz.$$

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We can now safely extend our notions to functions of several complex variables. The notion can be extended in two ways. Firstly, one could consider a function of several complex variables

$$\mathbb{C}^n \to \mathbb{C}.$$

Secondly, functions taking on values actually in $\mathbb{C}^n$. We must first consider what the correct basis choice would be for higher (complex) dimensions. As a basis for the topology in higher dimensions we will use these *polydisks*

$$B_\epsilon(w) = \{z : |z_j - w_j| < \epsilon_j\},$$

where $\epsilon := \{\epsilon_1, ..., \epsilon_n\}$.

Definition 1.1 Let $U \subset \mathbb{C}$. Let

$$f : U \to \mathbb{C}$$

be continuously differentiable. Then f is said to be *holomorphic* if the Cauchy-Riemann equations hold for all coordinates

$$z_j = x_j + iy_j.$$

I.e.,

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}$$

and

$$\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$$

for $j \in \{1, 2, ..., n\}$.

So by definition a continuously differentiable function f is holomorphic if the induced functions

$$f|_U : U \cap \{(z_1, z_2, ..., z_{j-1}, z, z_{j+1}, ..., z_n) : z \in \mathbb{C}\} \to \mathbb{C}$$

are holomorpic for every choice of j and for fixed $z_1, z_2, ..., z_{j-1}, z, z_{j+1}, ..., z_n \in \mathbb{C}$.

More simply put, a functions holomorphicity depends heavily on its component-wise holomorphicity as j ranges from 1 to n about each point $z \in \mathbb{C}$.

We can now rewrite our differential operators component-wise as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

and

$$\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

By the Cauchy-Riemann equations we have that

$$\frac{\partial f}{\partial \overline{z_j}} = 0$$

for $j \in \{1, 2, ..., n\}$. One tricky observation is that the equations in Definition 1.1 yield

$$\overline{\partial} f = 0.$$

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We next turn our attention to the Cauchy integral formula for functions of several variables. Let us first Recall what this means in $\mathbb{C}$:

Cauchy's Integral Formula (for $\mathbb{C}$) Let $U \subset \mathbb{C}$ and

$$f : U \to \mathbb{C}$$

holomorphic. Let γ is a closed curve in U. That is,

$$\gamma : [a, b] \to \mathbb{C}$$

is a continuous map with

$$\gamma(a) = \gamma(b).$$

Suppose $\eta(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus U$, then for $z_0 \in U \setminus \gamma([a, b])$ we have

$$\int_\gamma \frac{f(z)}{z - z_0} dz = (2\pi i)\eta(\gamma; z_0) f(z_0)$$

where

$$\eta(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}dz \in \mathbb{Z},$$

is the *winding number* of γ about z_0. Thus if the winding number about a particular point is 0, the value of the integral is therefore 0 as well.

Cauchys Integral Formula for ($\mathbb{C}^n$) Let

$$f\overline{B_\epsilon(w)} \to \mathbb{C}$$

be a continuous function such that f is holomorphic with respect to each component z_j in any point $z \in B_\epsilon(w)$. Then for any $z \in B_\epsilon(w)$,

$$\int_{|\xi_j - w_j| = \epsilon_j} \frac{f(\xi_1, \xi_2, ..., \xi_n)}{(\xi_1 - z_1)(\xi_2 - z_2)...(\xi_n - z_n)} d\xi_1 d\xi_2 \cdots d\xi_n = (2\pi i)^n f(z).$$

Proof. Repeated application of the Cauchy integral formula for the one variable case allows us to swap the iterated integral for the multiple integral. □

Thus continuous functions on open domains which are holomorphic with respect to each single coordinate (whilst others remain fixed) need be holomorphic! This is more famously known as Osgood's Lemma. The lemma is the specific case of Hartog's Theorem which we will get to momentarily. This result drops the assumption that the function need be continuous). As in the case for $\mathbb{C}$, the above integral can be used in writing out a power series expansion of any holomorphic function $f : U \to \mathbb{C}$. More precisely, for any $w \in U$, there exists a polydisk

$$B_\epsilon(w) \subset U \subset \mathbb{C}^n$$

such that the restriction $f|_{B_\epsilon(w)}$ is given via the power series

$$\sum_{j_1,\dots,j_n=0}^{\infty} a_{j_1,j_2,\dots,j_n}(z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

with coefficients given via

$$a_{j_1,\dots,j_n} = \frac{1}{j_1! \cdots j_n!} \cdot \frac{\partial^{j_1+\dots+j_n} f}{\partial z_1^{j_1} \cdots \partial z_n^{j_n}}.$$

From our above list of fun facts and theorems, the maximum principle, identity theorem, and Liouville generalize rather nicely into higher dimensions. A particular version of the Riemann Extension Theorem holds true and requires proof. The Riemann mapping theorem fails in higher dimensions however. To see the later, consider the the unit disk in $\mathbb{C}^2$ and the polydisk

$$B_{(1,1)}(0) \subset \mathbb{C}^2$$

which are not biholomorphic to one another. The proof is left as an exercise.

Exercise 3. Show the polydiskand the unit disk are not equivalent. That is, show there does not exits a biholomorphic map between

$$B_n := \{z \in \mathbb{C}^n : ||z|| < 1\}$$

and

$$D_n := \{z \in \mathbb{C}^n : |z_j| < 1\},$$

where $j = 1, 2, ..., n$.

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Often times the holomorphicity of a function of several variables is shown by representing a function as an integral of a known holomorphic function. The following Lemma will be of use to use further down the line.

> **Lemma 1.2** Let $U \subset \mathbb{C}^n$. Also let $V \subset \mathbb{C}$ be an open neighborhood of the boundary of $B_\epsilon(0) \subset \mathbb{C}$. Assume that
>
> $$f : V \times U \to \mathbb{C}$$
>
> is a holomorphic function. Then
>
> $$g(z) := g(z_1, ..., z_n) := \int_{|\zeta|=\epsilon} f(\xi, z_1, ..., z_n) d\xi$$
>
> is holomorphic on U.

Proof. Let $z \in U$. If $|\xi| = \epsilon$ then there exists a polydisk

$$B_{\delta(\xi)}(\xi) \times B_{\delta'(\xi)}(z) \subset V \times U$$

on which f has a power series expansion. As $\partial B_\epsilon(0)$ is compact we can find a finite number of points, call them

$$\xi_1, ..., \xi_n \in \partial B_\epsilon(0)$$

and positive real numbers

$$\delta(\xi_1), ..., \delta(\xi_n)$$

such that

$$\bigcup(\partial B_\epsilon(0) \cap B_{\frac{\delta(\xi_j)}{2}}(\xi_j))$$

is a disjoint union and

$$\partial B_\epsilon(0) = \bigcup(\partial B_\epsilon(0) \cap \overline{B_{\frac{\delta(\xi_j)}{2}}(\xi_j)}).$$

Hence,

$$\begin{aligned} g(z) &= \int_{|\xi|=\epsilon} f(\xi, z_1, ..., z_n) d\xi \\ &= \sum_{j=1}^{k} \int_{|\xi|=\epsilon, |\xi_j - \xi| < \frac{\delta(\xi_j)}{2}} f d\xi. \end{aligned}$$

Thus each summand is holomorphic since the power series expansion of f converges uniformly on

$$\overline{B_{\frac{\delta(\xi_j)}{2}}(\xi_j)}$$

and thus can be swapped with the integral. $\square$

The next result is a key result in the existence of a holomorphic extension without the assumption the function in question is continuous. Also note this result is only valid in $\mathbb{C}^n$ for $n \geq 2$.

Proposition 1.3: Hartog's Theorem Let $\epsilon = (\epsilon_1, ..., \epsilon_n)$, $\epsilon' = (\epsilon'_1, ..., \epsilon'_n)$be given such that for each $1 \leq j \leq n$ we have

$$\epsilon'_j < \epsilon_j.$$

If $n > 1$, then any holomorphic map

$$f : B_\epsilon(0) \setminus \overline{B_{\epsilon'(0)}} \to \mathbb{C}$$

can uniquely be extended to a holomorphic map

$$f : B_\epsilon(0) \to \mathbb{C}.$$

Proof. It suffices to assume $\epsilon = (1, ..., 1)$. That is, $B_\epsilon(0) \subset \mathbb{C}^n$ is the unit polydisk. As $\epsilon'_j < \epsilon_j$ for each j, there exists $\delta > 0$ such that the open set

$$V := \{z \in \mathbb{C}^n : 1 - \delta < |z_1| < 1, |z_{j \neq 1}| < 1\} \cup \{z \in \mathbb{C}^n : 1 - \delta < |z_2|, |z_{j \neq 2}| < 1\}$$

is properly contained in the complement

$$B_\epsilon(0) \setminus B_{\epsilon'}(0).$$

Note that this gives us an annulus "in between" z_1, z_2. In particular,

$$f : V \to \mathbb{C}$$

is holomorphic. Thus, for any

$$w := (z_2, ..., z_n) \in \mathbb{C}^{n-1},$$

with $|z_j| < 1$, we are guaranteed the existence of a holomorphic function

$$f_w(z_1) := f(z_1, z_2, ..., z_n) \in \mathbb{C}^n$$

on

$$\{z \in \mathbb{C}^n : 1 - \delta < |z_1| < 1, |z_{j \neq 1}| < 1\}.$$

Let

$$f_w(z_1) := \sum_{n=-\infty}^{\infty} a_n(w) z_1^n$$

be the Laurent series of this function. Then the coefficients are given by

$$a_n(w) = \frac{1}{2\pi i} \int_{|\xi| = 1 - \frac{\delta}{2}} \frac{f_w(\xi)}{\xi^{n+1}} d\xi$$

(this is a result of the Laurent series development) which is holomorphic for w in the unit polydisk of $\mathbb{C}^{n-1}$ by our Lemma. So for a fixed $w \in \mathbb{C}^{n-1}$, the map given via

$$z_1 \mapsto f_w(z_1)$$

is holomorphic on the unit polydisk such that

$$1 - \delta < |z_2| < 1.$$

Thus the negative coefficients are all zero. This forces the coefficients to be identically zero. We can now extend f holomorphically to $\overline{f}$ by the following power series:

$$\sum_{n=0}^{\infty} a_n(w) z_1^n.$$

As the $a_n(w)$ are holomorphic, they attain their max on the boundary. Thus convergence on our annulus yields uniform convergence everywhere. The holomorphic function given in our series glues together with f to give us the desired holomorphism. □

It should be noted that our previous result fails in the one variable case. Informally, Hartog's Theorem says the singularities of f are given by vanishing sets of holomorphic functions.

Next we will examine the Weierstrass preparation theorem (WPT), which will play a key role in the theory of functions of several complex variables.

Let

$$f : B_\epsilon(0) \to \mathbb{C}$$

be holomorphic on the polydisk $B_\epsilon(0)$. For any $w = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$ we let $f_w(z_1)$ denote the function $f(z_1, ..., z_n)$ over $\mathbb{C}^n$. We show that the zeroes of f are caused by a factor of f which has the form of a Weierstrass polynomial.

Definition 1.5 A *Weierstrass polynomial* in z_1 of the form

$$\sum_{j=0}^{d} \alpha_j(w) z_1^{d-j}$$

where the $a_j(w)$ are each holomorphic functions on some small polydisk in $\mathbb{C}^{n-1}$ such that they vanish at the origin.

Recall that any holomorphic function of one complex variable with a zero of order d can be written as

$$z^d h(z),$$

with $h(0) \neq 0$.

Note a zero of order d of $f_0(z_1)$ could decompose into a collection of zeroes of $f_w(z_1)$ whose orders ass up to d. The following generalizes the notion of local normal form for the one variable case.

Proposition 1.4: Weierstrass Prep Theorem Let

$$f : B_\epsilon(0) \to \mathbb{C}$$

be holomorphic on the polydisk $B_\epsilon(0)$. Assume $f(0) = 0$ and $f_0(z_1) \neq 0$. Then there exists a (unique) Weierstrass Polynomial

$$g(z_1, w) = g_w(z_1)$$

and a holomorphic function h on some smaller polydisk $B_{\epsilon'}(0) \subset B_\epsilon(0)$ such that

$$f = g \cdot h$$

and $h(0) \neq 0$.

Proof. Let f be given. By the fundamental theorem, we know f has zeroes over the complex number. Thus we can let the multiplicity of f be d with zeroes

$$a_1(w), ..., a_d(w).$$

Since f_0 is not identically zero, we can find an $\epsilon_1 > 0$ such that $f_0 \in \overline{B_{\epsilon_1}(0)}$ vanishes only in 0. Then choose $\epsilon_2, ..., \epsilon_n > 0$ such that

$$f(z_1, z_2, ..., z_n) \neq 0$$

for $|z_1| = \epsilon_1$ and $|z_j| < \epsilon_j$ for $j = 2, ..., n$. Note if $w = 0$ then

$$a_1(0) = ... = a_d(0) = 0.$$

Here each zero occurs as much as its multiplicity determines. Next, it would be nice if we knew if there was a relationship between d and w. That is, if d depends on w or not. To see this, consider following polynomial (Which has the same zeroes, with multiplicities as $f_w(z_1)$),

$$g_w(z_1) := \prod_{j=1}^{d} (z_1 - a_j(w)).$$

Thus, for a fixed w, the function

$$h_w(z_1) := \frac{f_w(z_1)}{g_w(z_1)}$$

is holomorphic in z_1. (As quotients are holomorphic provided $g'(z_1) \neq 0$). We must now only show $g_w(z_1), h_w(z_1)$ are holomorphic in w. To see this, note that the coefficients of $g_w(z_1)$ can be written as

$$\sum_{j=1}^{d} a_j(w)^k$$

for $k = 1, 2, ..., n$. Thus

$$g_w(z_1) - w(z_1) = \prod_{j=1}^{d} (z_1 - a_j(w)) - w(z_1)$$

is holomorphic in w given the above summands are holomorphic in w. We must apply the Residue Theorem to

$$z_1^k \frac{f'_w(z_1)}{f_w(z_1)}.$$

Let

$$f_w(\xi) = \sum_{j=m}^{\infty} a_j (\xi - a)^j$$

be the power series of f_w in some zero, a. Then it is clear to see that the derivative is given via

$$f_w'(\xi) = \sum_{j=m}^{\infty} j a_j (\xi - a)^{j-1}.$$

Moreover,

$$\xi^k = a^k + k a^{k-1}(\xi - a) + ...$$

Then from a first the residue theorem we get

$$\mathrm{Res}_{\xi=a}(\xi^k \frac{f_w'(\xi)}{f_w(\xi)}) = m a^k.$$

Therefore, the polynomial expression og $g_w(z_1)$ can be written as

$$\sum_{j=1}^{d} a_j(w)^k = \frac{1}{2\pi i} \int_{|\xi|=\epsilon_1} \xi^k \frac{f_w'(\xi)}{f_w(\xi)} d\xi.$$

The left is holomorphic in w by our Lemma as f_w is holomorphic in w, thus $g_w(z_1)$ is holomorphic in $z_1, ..., z_n$. Note that when $k = 0$, the left side is the number of zeroes of f_w counted with multiplicities. Thus this integer d depends holomorphically on w and therefore does not depend on w at all.

Observe that

$$\{(z_1, w) : z_1 = a_j(w)\}^c$$

for some j, contains a neighborhood of

$$\{(z_1, w) : |z_1| = \epsilon_1, |z_{j\neq 1}| < \epsilon_j\}.$$

And so by the Cauchy integral formula

$$h_w(z_1) = \frac{1}{2\pi i} \int_{|\xi|=\epsilon_1} \frac{h_w(\xi)}{(\xi - z_1)} d\xi,$$

together with the holomorphicity of $\frac{f}{g}$ gives us that h is holomorphic everywhere, by our lemma.

For the uniqueness, since $h(0) \neq 0$, h does not vanish anywhere thus f_w, g_w have the same vanishing sets and the only Weierstrass polynomial with this property is polynomial we have just constructed. □

From here on, as a short hand we will let $Z(f)$ denote the *zero set* or vanishing set of a holomorphic function f. That is,

$$Z(f) = \{z : f(z) = 0\}.$$

Proposition 1.7 (Riemann extension theorem) Let f be holomorphic on some $U \subset \mathbb{C}^n$. If

$$g : U \setminus Z(f) \to \mathbb{C}$$

is holomorphic and locally bounded near $Z(f)$, then g can be uniquely extended to a holomorphic function

$$\overline{g} : U \to \mathbb{C}.$$

Proof. First we examine the special case for when $n = 2$ and $f(z) = z_1$. Then

$$g_{z_2}(z_1) := g(z_1, z_2)$$

is bounded and holomorphic on some punctured disk in the plane. That is, bounded and holomorphic near $Z(f)$. Thus we can find an extension of g_{z_2} to a holomorphic function on the whole disk. It would be left to show these functions all glue together in the piece-wise sense.

For $n \geq 3$, we may very well suppose that $U \subset \mathbb{C}$ is given by

$$U := B_\epsilon(0).$$

Let us define the line in $\mathbb{C}^n$ via

$$\mathcal{L}_1 := \{(z_1, 0, ..., 0) : z_1 \in \mathbb{C}\}$$

Note: $\mathcal{L}_j$ *makes all coordinates zero other than* z_j

Thus we may assume that

$$U \cap L \nsubseteq Z(f)$$

. Additionally we can even boil this down to the case that the restriction f_0 of f to $\mathcal{L}$ vanishes only at $(0, 0, ..., 0)$. Thus for when we set

$$|z_1| = \frac{\epsilon_1}{2}$$

we get that

$$f_0(z_1) \neq 0.$$

Then we could wiggle our $\epsilon_2, ..., \epsilon_n > 0$ in order to assume

$$f(z) \neq 0$$

given that

$$|z_1| = \frac{\epsilon_1}{2},$$

and

$$|z_{j \neq 1}| < \frac{\epsilon_j}{2}.$$

That is, for any given w with the above given condition above on the coordinates of w, the function f_w has no zeroes on

$$\partial B_{\frac{\epsilon_1}{2}}(0).$$

(Must recall what it means having no zeroes on the boundary By our assumption, the restriction g_w of g to $B_{\frac{\epsilon_1}{2}}(0) \setminus Z(f_w)$ is bounded and thus can be extended to a holomorphic function $\tilde{g}$ on $B_{\frac{\epsilon_1}{2}}(0)$. By Cauchy's infamous integral formula however, this extension is given via

$$\tilde{g}_w(z_1) = \frac{1}{2\pi i} \int_{\partial B_{\frac{\epsilon_1}{2}}(0)} \frac{g_w(\xi)}{\xi - z_1} d\xi.$$

As f_x has no zeroes on this boundary,

$$f_w(\xi) \neq 0$$

for any $\xi \in \partial B_{\frac{\epsilon_1}{2}}(0)$. In turn, the integrand is holomorphic in (z_1, w). By our Lemma 1.3 (once again),

$$\tilde{g}(z_1, w) := \tilde{g}_w(z_1)$$

is then holomorphic on (z_1, w) and we thus have the holomorphic of extension of g, namely this $\tilde{g}_w(z_1)$. □

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We now focus our attention to extend the notion of holomorphicity to function who take on values in $\mathbb{C}^n$. We do so as follows.

Definition 1.8 Let $U \subset \mathbb{C}^m$ be open.Then a map

$$f : U \to \mathbb{C}^n$$

is said to be *holomorphic* if each coordinate function $f_1, f_2, ..., f_n$ is a holomorphic function

$$f_j : U \to \mathbb{C}.$$

In an analogy to the one-dimensional case, we say

$$f : U \to V$$

where $U, V \subset \mathbb{C}^n$ is *biholomorphic* if and only if f is a bijective, holomorphic, with a holomorphic inverse.

Definition 1.8 Let $U \subset \mathbb{C}^m$ be open and let

$$f : U \to \mathbb{C}^n$$

be holomorphic. Then the (complex) Jacobian of f at some $z \in U$ is the matrix given via

(Text not finished yet, work in progress for a while. This is my first attempt at a text book or informal notes on a given topic).

2. Complex and Hermitian structures

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