Integration on Riemann surfaces: An intro to Differential forms
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Texts used:
Algebraic Curves and Riemann Surfaces by Rick Miranda
Calculus on Manifolds by Michael Spivak
MyMathYourMath.com

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Tensors

§ 1.1 Tensor Definitions

Definition: If V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, we denote the k fold product V^k as k products of V. A function

$$T : V^k \to \mathbb{R}$$

is called *multilinear* if for each j between 1 and k,

$$T(v_1, ..., v_j + v_j', ..., v_k) = T(v_1, ..., v_j, ..., v_k) + T(v_1, ..., v_j', ..., v_k)$$

and

$$T(v_1, ..., av_j', ..., v_k) = aT(v_1, ..., v_j', ..., v_k).$$

A multi-linear function is called a $k-$tensor on V and the set of all $k-$tensors becomes a vector space we will denote by $\mathcal{T}^k(V)$. So for $S, T \in \mathcal{T}^k(V)$ and $a \in \mathcal{F}$ some field (usually either the reals or complex), one has

$$(S+T)(\overline{v}) = S(\overline{v}) + T(\overline{v})$$

and

$$(aS)(\overline{v}) = a \cdot S(\overline{v}).$$

Here $\overline{v} = (v_1, v_2, ..., v_k) \in V^k$. Naturally, one would like some well-defined operation for members of difference dimension vector spaces of tensors.

Definition: Let's say that $S \in \mathcal{T}^k(V), T \in \mathcal{T}^l(V)$. Define the *tensor product* $S \otimes T \in \mathcal{T}^{k+l}(V)$ via

$$S \otimes T(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) := S(v_1, ..., v_k) \cdot T(v_{k+1}, ..., v_l)$$

It is worth noting the operation $\otimes$ is non-commutative. It is left to the reader to verify that

$$\begin{aligned}
(S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T \\
S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2 \\
(aS) \otimes T &= S \otimes (sT) = a(S \otimes T) \\
(S \otimes T) \otimes U &= S \otimes (T \otimes U)
\end{aligned}$$

For the last property, note that the associativity will holds for any finite length cross product. Note that for $k = 1$, $\mathcal{T}^1(V)$ is the dual space V^*. Furthermore, our operation $\otimes$ allows us to express $\mathcal{T}^k(V)$ in terms of $\mathcal{T}^1(V)$. That is, 1-forms can be used to build up higher dimensional forms. Also we have yet to define what a "form" is so hang tight.

A similar construction for dual spaces can be made for tensors.

Definition: If

$$F : V \to W$$

is a linear transformation, then *tensor induced linear transformation* is given as

$$f^* : \mathcal{T}^k(W) \to \mathcal{T}^k(V)$$

and is defined via

$$f^*(T(\overline{v})) = T(f(\overline{v})).$$

One can easily verify using properties of $\otimes$ that

$$f^*(S \otimes T) = f^*S \otimes f^*T.$$

One example of tensors is the inner product operation $\langle\rangle \in \mathcal{T}^2(\mathbb{R}^n)$.

Note that the determinant is also a tensor. Namely $\det \in \mathcal{T}^n(\mathbb{R}^n)$. Recall that when interchanging two rows whilst keeping all else fixed results in a determinant with opposite sign. This leads us to our next definition.

Definition: A $k-$tensor $\omega \in \mathcal{T}^k(V)$ is called an *alternating tensor* if

$$\omega(v_1, ..., v_i, .., v_j, ..., v_k) = -\omega(v_1, ..., v_j, .., v_i, ..., v_k)$$

Observe that the set of alternating $k-$tensors is a subspace $\Lambda^k(V)$ of $\mathcal{T}^k(V)$. So then if $T \in \mathcal{T}^{(}V)$, then define

$$\text{Alt}(T)(v_1, ..., v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}\sigma \cdot T(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

where S_k is the permutation group on k elements. It can be left as an exercise to the reader to verify that

1. If $T \in \mathcal{T}^k(V)$, then $\text{Alt}(T) \in \Lambda^k(V)$.
2. If $\omega \in \Lambda^k(V)$, then $\text{Alt}\omega = \omega$.
3. If $T \in \mathcal{T}^k(V)$, then $\text{Alt}(\text{Alt}(T)) = \text{Alt}(T)$.

§ **1.2 Wedge Product**

The issue with the operation $\otimes$ is that in the subspace of alternating tensors, it need not be closed.

Definition: We therefore define a new product $\wedge$ called the *wedge product*.That is if $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V)$, then $\omega \wedge \eta \in \Lambda^{k+l}(V)$ is defined via

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\text{Alt}(\omega \otimes \eta).$$

The wedge product $\wedge$ comes with the following properties:

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Holomorphic and Meromorphic 1-forms

From one-variable Complex Analysis recall that the method of integrating was using the contour integration. We focus our attention to a concept known as *differential forms*. This is the concept required to integrate over a manifold such as a Riemann surface.

§2.1 Holomorphic 1-Forms

Definition: A *holomorphic 1-form* on an open subset $U \subseteq \mathbb{C}$ is an expression ω of the form

$$\omega = f(z)\mathrm{d}z.$$

where f is holomorphic on U. We say ω is a holomorphic 1-form in *the coordinate* z. (It should be noted that constant functions are $0-$forms.) We would like to transfer this notion over to integrating over a given Riemann surface. We require a condition on domains that overlap. This notion will be made rigorous in the following definition.

Definition: Suppose that $\omega_1 = f(z)\mathrm{d}z$ and $\omega_2 = g(w)\mathrm{d}w$ where f, g are holomorphic on $V_1, V_2 \subseteq \mathbb{C}$ respectively. Let

$$T : V_2 \to V_1$$

be defined via

$$z := T(w).$$

We say that ω_1 *transforms to* ω_2 *under* T if

$$g(w) = f(T(w)) \cdot T'(w).$$

Here, when this transformation is made one typically writes

$$\mathrm{d}z = T'(w)\mathrm{d}w.$$

If T is $1-1$, then there exists a function

$$S : V_1 \to V_2$$

such that S transforms ω_2 into ω_1 if and only if T sends ω_1 to ω_2. We can now transform this construction to Riemann surfaces.

Definition: Let X be a Riemann surface. A *holomorphic 1-form* on X is a collection of holomorphic 1-forms $\{\omega_\phi\}$, one for each chart

$$\phi : U \to V$$

in the coordinate of the target V such that if two charts

$$\phi_j : U_j \to V_j$$

for $j = 1, 2$ have domains which overlap, then the associated holomorphic 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinate mapping

$$T = \phi_1 \circ \phi_2^{-1}.$$

In order to define a holomorphic 1-form on a Riemann surface one only needs to give a holomorphic 1-form on the chart of certain atlases. This will be formalized rigorously as follows.

Lemma 1 *Let X be a Riemann surface and $\mathcal{A}$ a complex atlas on X. Suppose holomorphic 1-forms are given for each chart of this atlas, which transfer to each other on overlapping domains. Then there exists a unique holomorphic 1-form on X extending these holomorphic 1-forms on each of the charts of A.*

Proof. Let ψ be a chart of X not in $\mathcal{A}$. We need to define the holomorphic 1-form with respect to ψ with

$$\psi : U \to \mathbb{C},$$

or equivalently, in terms of our local coordinate w of ψ. Fix some $p \in U$. Choose a chart ϕ in $\mathcal{A}$ with

$$\phi : V \to \mathbb{C}$$

such that $p \in V$. Here, let z be the associated local variable. Let $f(z)\mathrm{d}z$ be the holomorphic 1-form with respect to ϕ. Then we can use the transformation property to take the holomorphic 1-form of ψ to be

$$g(w) := f(T(w))T'(w).$$

Here $z = T(w)$ described the change of coordinates $\phi \circ \psi^{-1}$. We must only check this definition is independent of our choice of ϕ and gives a 1-form with respect to ψ at every point in V. As long as all of these holomorphic 1-forms transform to each other in the sense we defined above, and thus define a holomorphic 1-form on X. This 1-form is then unique. □

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§2.2 Meromorphic 1-Forms

In the same fashion we may define a meromorphic 1-form.

Definition: A *meromorphic* $1-$*form* on an open subset $U \subseteq \mathbb{C}$ is an expression ω of the form

$$\omega = f(z)\mathrm{d}z,$$

where f is meromorphic on U. We say that ω is a mermomorphic $1-$form in *the coordinate* z. Compatibility definitions agree with those of holomorphic 1-forms. This is formulated as follows.

Definition: Suppose that $\omega_1 = f(z)\mathrm{d}z$ and $\omega_2 = g(w)\mathrm{d}w$ where f, g are meromorphic on $V_1, V_2 \subseteq \mathbb{C}$ respectively. Let

$$z = T(w)$$

define a holomorphic mapping from V_2 into V_1. We say ω_1 *transforms to* ω_2 *under* T if

$$g(w) = f(T(w))T'(w).$$

Transforming the notion of meromophic 1-forms from the plane to a Riemann surface is done similarly as for holomorphic functions.

Definition: Let X be a Riemann surface. A *meromorphic* $1-$*form* on X is a collection $\{\omega_\phi\}$ of meromorphic 1-forms, one for each chart

$$\phi : U \to V$$

in the variable of the target V, such that if two charts

$$\phi_j : U_j \to V_j,$$

for $j = 1, 2$, have overlapping domains, then the associated meromorphic 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinate mapping

$$T = \phi_1 \circ \phi_2^{-1}.$$

This leads us to our next lemma.

Lemma 2 *Let X be a Riemann surface and $\mathcal{A}$ a complex atlas on X. Suppose that meromorphic 1-forms are given for each chart of $\mathcal{A}$, which transfer each other on their domains. Then there exists a unique meromorphic 1-form on X extending these meromorphic 1-forms on each of the charts of $\mathcal{A}$.*

Let ω be a meromorphic 1-form defined in a neighborhood of some point p. If we choose a local coordinate chart centered at p, we can write

$$\omega = f(z)\mathrm{d}z$$

where f is meromorphic at $z = 0$.

Definition: The *order of ω at p*, denoted by $\mathrm{ord}_p(\omega)$ is the order of the function at $z = 0$.

Note that $\mathrm{ord}_p(\omega)$ is well-defined and is independent of choice of coordinate chart. A meromorphic 1-form ω is holomorphic at p if and only if $\mathrm{ord}_p(\omega) \geq 0$. We say p is a *zero of ω of order n* if the order is strictly positive. We say p is a *pole of ω of order n* if $\mathrm{ord}_p(\omega) = -n < 0$. Furthermore, note the set of zeroes and poles of a meromorphic functions forms a discrete set as apposed to continuous.

§2.3 Defining Meromorphic Functions and Forms with a Formula

The definitions of forms on holomorphic or meromorphic functions implies that in order to define ω on some Riemann surface X, one must give local expressions for ω, in the form

$$f(z)\mathrm{d}z,$$

in each chart of an atlas $\mathcal{A}$. In face one can use a single formula in a single chart. This is enough to determine ω by the Identity theorem for meromorphic functions and forms: If two meromorphic 1-forms agree on some open domain, they must be identical.

Of course, this definition does not guarantee that the forms exists on all of X. One can have the case that given a meromorphic local expression of the form

$$f(x)\mathrm{d}z$$

in one chart, then when one transforms this local expression to another chart, the meromorphicity may fail to get carried over. I.e., the meromorphic 1-form

$$\exp(z)\mathrm{d}z$$

on the finite chart $\mathbb{C}$ of $\mathbb{C}_\infty$ does not extend meromorphically to a neighborhood of ∞.

Additionally, one may encounter the following issue: The local expression many not transform uniquely to all other points of X. I.e., consider the meromorphic 1-form

$$z^{\frac{1}{2}}\mathrm{d}z$$

defined on the plane with the negative real line removed, where the branch of the square root is chosen such that

$$\sqrt{1} = 1.$$

This can be extended to the negatives reals, but not in the unique sense. Thus there is no meromorphic function on the nonzero complex numbers.

However, it is convenient to use a single formula in one specified chart to define a meromorphic 1-form ω, and one can check that the formula transforms uniquely to meromorphic 1-form on all of X. Furthermore, meromorphic functions can be determined by a single chart as we have seen before.

§2.4 Conjugate of z

We can for now relax the meromorphic and holomorphic conditions for 1-forms and obtain this notion of C^∞ 1-forms. Locally, these should be the expressions

$$f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$$

where x, y are local real variables and

$$z = x + iy.$$

However we can completely use z and it's complex conjugate $\overline{z}$ instead of real and imaginary parts x, y. We can do this since we have

$$x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}.$$

Here

$$z = x + iy, \overline{z} = x - iy,$$

such that any function expressible in terms of x, y is expressible in terms of $z, \overline{z}$ and vis versa. Note then that since

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y, \mathrm{d}\overline{z} = \mathrm{d}x - i\mathrm{d}y.$$

We have the differentials

$$\mathrm{d}x = \frac{\mathrm{d}z + \mathrm{d}\overline{z}}{2}, \mathrm{d}y = \frac{\mathrm{d}z - \mathrm{d}\overline{z}}{2i}.$$

And so any expression one would like to construct of the form

$$f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y,$$

can instead be written as

$$u(z,\overline{z})\mathrm{d}z + v(z,\overline{z})\mathrm{d}\overline{z}.$$

We can formulate this a bit more rigorously as follows. We carry the principle on over to partial derivatives as well. Given a C^∞ function $f(x, y)$, we have that

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \\ &= \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} \\ &= \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}. \end{aligned}$$

We can now define the differential operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \overline{z}}$ via

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With this given notation we require a C^∞ function f to be holomorphic on some open subset $U \subseteq \mathbb{C}$ if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

since this is equivalent to the Cauchy-Riemann equations for f.

Definition: A C^∞ $1-$*form* on an open subset $U \subseteq \mathbb{C}$ is an expression ω of the form

$$\omega = f(z, \overline{z})\mathrm{d}z + g(z, \overline{z})\mathrm{d}\overline{z},$$

where f, g are C^∞ functions on U. We say that ω is a C^∞ 1-form *in the coordinate* z. This comes with the following transformation rule.

Definition: Suppose that $\omega_1 = f_1(z, \overline{z})\mathrm{d}z + g_1(z, \overline{z})\mathrm{d}\overline{z}$ is a C^∞ 1-form in the coordinate z, defined on some open $U_1 \subseteq \mathbb{C}$. Also suppose that $\omega_2 = f_2(w, \overline{w})\mathrm{d}w + g_2(w, \overline{w})\mathrm{d}\overline{w}$ is a C^∞ 1-form in the coordinate $z$is a C^∞ 1-form in the coordinate w, defined on some open $U_2 \subseteq \mathbb{C}$. Let $z = T(w)$ define a holomorphic mapping

$$V_2 \to V_1.$$

We say that ω_1 *transforms to* ω_2 *under* T if

$$f_2(w, \overline{w}) = f_1(T(w), \overline{T(w)})T'(w),$$

and

$$g_2(w, \overline{w}) = g_1(T(w), \overline{T(w)})T'(w).$$

Note that the definition is made in this fashion by differential formula for the chain rule. That is, if $z = T(w)$, then $\mathrm{d}z = T'(w)\mathrm{d}w$ and $\mathrm{d}\overline{z} = \overline{T'(w)}\mathrm{d}\overline{z}$. Furthermore, note that the $\mathrm{d}z$ part transforms into $\mathrm{d}w$ and the $\mathrm{d}\overline{z}$ into the $\mathrm{d}\overline{w}$: there is no "mixing" of the two halves of the expression upon a change of coordinates. This is the real reason we use z and its conjugate as apposed to x, y. We use the same method as before to transport this notion to a Riemann surface.

Definition: Let X be a Riemann surface. A C^∞ *1-form on X* is a collection of C^∞ 1-forms $\{\omega_\phi\}$, one for each chart $\phi : U \to V$ in the variable of the target V. such that if two charts

$$\phi_j : U_j \to V_i$$

for $j = 1, 2$, have overlapping domains, then the associated C^∞ 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinate mapping

$$T = \phi_1 \circ \phi_2^{-1}.$$

This leads us to another lemma.

Lemma 3 *Let X be a Riemann surface and $\mathcal{A}$ a complex atlas on X. Suppose that C^∞ 1-forms are given for each chart of $\mathcal{A}$, which transform to each other on their common domains. Then there exists a unique C^∞ 1-form on X extending these C^∞ 1-forms on each of the charts of $\mathcal{A}$.*

The next section is devoted to *types* of 1-forms.

§2.5 Types of 1-forms

Since under transformation by holomorphic change of coordinates, the $\mathrm{d}z$ and $\mathrm{d}\overline{z}$ of the C^∞ 1-form are preserved, we may split the definitions into two separate definitions, namely of C^∞ 1-forms with only $\mathrm{d}z$ parts, and ones with only $\mathrm{d}\overline{z}$ parts.

Definition: A C^∞ 1-form is of *type* $(1, 0)$ if it is locally of the form $f(z, \overline{z})\mathrm{d}z$. It is of *type* $(0, 1)$ if it is locally of the form $g(z, \overline{z})\mathrm{d}\overline{z}$.

Note that the type is preserved over domains that overlap. Furthermore, note any holomorphic 1-form is of type (1,0). A meromorphic 1-form would be of this type except it is not C^∞ at its poles.

§2.6 C^∞ 2-forms

One typically introduces 1-forms in order to integrate along a line or a path in space. Similarly one would like to integrate over a 2-dimensional piece of a Riemann surface. The integrand in this case would be a 2-form.

Definition: A C^∞ *2-form* on an open subset $U \subseteq \mathbb{C}$ is an expression η of the form

$$\eta = f(z, \overline{z})\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $f \in C^\infty(U)$.

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Operations on forms

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