Math 562 - Complex Analysis I Notes
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California State University of Long Beach

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Text used: Functions of One Complex Variable I by John B. Conway
mymathyourmath.com

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Abstract

In these notes, we will give a rather brief overview at an introductory course on one complex variable, at the Masters level. Prerequisites for these notes include, but are not limited to, prior knowledge of basic Point-Set Topology, basic group and ring theory, a course in undergraduate real analysis. No graduate level knowledge is required for the understanding of these notes. We start off with basic properties and move onto studying structural behaviours of analytic functions defined one open subsets of $\mathbb{C}$.

Contents

Lecture 1 ; August 26, 2019

(I) Definitions

Define

$$\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(x, y)|x, y \in \mathbb{R}\}$$

to be the set of complex numbers. One can endow $\mathbb{C}$ with the natural operations $+$ and $\cdot$ defined via

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

$\forall a, b, c, d \in \mathbb{R}$. This turns $(\mathbb{C}, +, \cdot)$ a field with additive identity and multiplicative identities

$$+_e = (0, 0)$$

$$\cdot_e = (1, 0)$$

(II) $\mathbb{C}$ as a Field

Since we know $(\mathbb{R}, +, \cdot)$ is a field, there is a natural field homomorphism

$$F : (\mathbb{R}, +, \cdot) \to (\mathbb{C}, +, \cdot)$$

defined via

$$F(x) = (x, 0)$$

Note F is 1-1 so it also has an inverse. Thus, it is natural to write $(a, 0)$ for some $a \in \mathbb{R}$. Furthermore, we know for some $(a, b) \in \mathbb{C}$ that

$$(a, b) = (a, 0) + (0, b) = a + b(0, 1),$$

where $(0, 1) := i$, thus we it makes sense to write

$$\mathbb{C} = \{a + bi|a, b \in \mathbb{R}\}.$$

It is left as an exercise to the reader to show why $i = \sqrt{-1}$. Note that $a + bi = c + di \Leftrightarrow a = c$ and $b = d$. If we write

$$z = a + bi,$$

then one can define

$$\mathrm{Re}(z) = a, \mathrm{Im}(z) = b.$$

(III) $\mathbb{C}$ as a Metric Space

Turns out $\mathbb{C}$ can be turned into a metric space First we let

$$z = a + bi, w = c + di$$

Then let $d(z, w)$ denote distance or the *metric function*:

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

(distnace between any two points). Note that d enjoys the Triangle Inequality. That is, for every $z, w, v \in \mathbb{C}$,

$$d(z, w) \leq d(z, v) + d(v, w)$$

For any $z \in \mathbb{C}$ define $|z|$ as distance from $(0, 0)$ to $z = a + bi$, given via

$$|z| = \sqrt{a^2 + b^2}$$

(distance between any one point and origin). Then by Triangle Inequality,

$$|w + z| \leq |w| + |z|.$$

And since $||$ is multiplicative, we have the equality

$$|zw| = |z||w|.$$

(IV) Complex Conjugate

Cool fact,

$$a^2 + b^2$$

is actually a difference of squares in $\mathbb{C}$, to see this just let $z, w \in \mathbb{C}$, then

$$z^2 + w^2 = (z + w)(z - w).$$

This is clearly not true in $\mathbb{R}$. To make some sense of this, we define

$$\overline{z} := a - bi \quad ; \quad z = a + bi \in \mathbb{C}$$

as the *complex conjugate* of $z \in \mathbb{C}$. The complex conjugate enjoys some nice properties:

(a) $z\overline{z} = |z|^2$

(b) $|z| = |\overline{z}|$

(c) $z + \overline{z} = 2\text{Re}(z)$

(V) Inequality Theory

A binary relation $\leq$ is a *total ordering* on a set X if the following holds

$$1. \forall a, b \in X a \leq b \bigvee b \leq a \quad (connexity)$$

$$2. \forall a, b, c \in X, (a \leq b \bigwedge b \leq c) \Rightarrow a \leq c \quad (transitivity)$$

$$3. \forall a, b, c \in X (a \leq b \bigwedge c \leq 0) \Rightarrow ac \leq bc \quad (anti - symetry).$$

Guess what? $\mathbb{C}$ is not totally ordered.

Proof. Suppose not, suppose there is some total ordering, $\succeq$, on $\mathbb{C}$. Then $\succeq$ satisfies $(1), (2)$ and (3). We start by letting

$$i \succeq 0.$$

Multiply both side by an i and use $i^2 = -1$ to obtain

$$-1 \succeq 0,$$

add 1 to both sides and we get

$$0 \succeq 1.$$

Multiply both sides again by an i and we have that

$$0 \succeq i$$

Thus we have

$$(i \succeq 0) \bigwedge (0 \succeq i)$$

which forces

$$0 = i,$$

a contradiction therefore $\mathbb{C}$ has no total ordering. □

(VI) Polar Coordinates

Let $z \in \mathbb{C}$. One could find alternate (canonical) form for z other than rectangular as follows:

$$z = x + yi \quad ; \quad x, y \in \mathbb{R}$$

We could let z be represented by the ordered pair

$$(r, \theta) \quad ; \quad r = |z|$$

And $\theta \in \arg z$; where

$$\arg z = \{\theta + 2\pi n | n \in \mathbb{Z}\}.$$

Polar coordinates give us a couple nice results:

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$(re^{i\theta})^n = r^n e^{in\theta},$$

where $e^{i\theta} = \cos\theta + i \sin\theta$. Note that

$$e^{2\pi i n} = 1$$

for every $n \in \mathbb{Z}$. Moreover we have that

$$e^{\pi i n} = -1$$

for every $n \in \mathbb{Z}$.

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Lecture 2 ; August 28, 2019

(I) Sequences and Series

Let $\{a_1, a_2, ...\}$ be a sequence in $\mathbb{C}$. We say the series $\sum_{j=1}^{n} a_j$ *converges* to A if for any given $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if

$$m \geq N$$

then

$$|\sum_{n=1}^{m} a_n - A| < \epsilon.$$

If in addition we have that $\sum |a_n|$ converges, we say the series *converges absolutely*. Define a sequence $\{a_n\}$ to be *Cauchy* if for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if

$$m, n \geq N$$

then,

$$|a_n - a_m| < \epsilon.$$

So naively speaking, the sequence converges "term-wise" as well. With this definition it is natural to state the following corollary:

> **Corollary:** A sequence converges if and only if it is a Cauchy sequence. We now define the *tail-end* of a given sequence via the following biconditional: The series $S_N = \sum_{n=1}^{N} a_n$ converges if and only if the sequence
>
> $$t_m = \sum_{n=m}^{\infty} a_n$$
>
> goes to 0 as $n \to \infty$.

Note that any subsequence of a convergent sequence always converges. We conclude series with the following key theorem in analysis.

> **Theorem 1**: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof. Let $\epsilon > 0$ be given and suppose $\sum |a_n|$ converges. Then we know the tail-end converges. That is,

$$t_m = \sum_{n=m}^{\infty} |a_n| \to 0$$

as $n \to \infty$. Let $N \in \mathbb{N}$ be such that if $m \geq N$, then

$$|\sum_{n=m}^{\infty} a_n - 0| = |\sum_{n=m}^{\infty} a_n| < \epsilon.$$

Consider the series

$$S_m = \sum_{n=1}^{m} a_n$$

I claim S_m is Cauchy. Let $m, k \geq N$, without any loss of generality let us assume $m > k$, then

$$\begin{aligned}
|S_m - S_k| &= |\sum_{k+1}^{m} a_n| \\
&\leq \sum_{n=k+1}^{m} |a_n| \\
&\leq \sum_{n=k+1}^{\infty} |a_n| \\
&< \epsilon.
\end{aligned}$$

by the tail end thus $\{S_m\}$ is Cauchy, hence convergent. □

Next, let $\{a_n\} \subseteq \mathbb{R}$ be a sequence. We *define*,

$$\limsup a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, ...\}$$

$$\liminf a_n = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, ...\}$$

Activity 1. Let $a_n = (-1)^n$, then

$$\limsup a_n = 1,$$

and

$$\liminf a_n = -1.$$

Activity 2. Let $a_n = ((-1)^n + 1)n$, then

$$a_n = \begin{cases} 0 & ; n \quad \text{odd} \\ 2n & ; n \quad \text{even} \end{cases}$$

Here, the reader can verify that

$$\limsup a_n = \infty, \qquad \liminf a_n = 0$$

(II) Power Series Intro A *power series* is an infinite series of the form

$$\sum_{n=1}^{\infty} a_n (z-a)^n,$$

where $\{a_n\} \subseteq \mathbb{C}$.
Example 1. The Geometric series

$$\sum_{n=0}^{\infty} z^n \to \frac{1}{1-z}; \qquad |z| < 1$$

we know *diverges* if $|z| > 1$.

(III) Power Series Big Theorem We start this section off with a big result.

Theorem 2: (Conway III.1.3) For a given power series centered about a,

$$\sum_{n=0}^{\infty} a_n(z-a)^n,$$

define, for $R \in [0,\infty]$,

$$\frac{1}{R} := \limsup |a_n|^{\frac{1}{n}}.$$

Then,

(a) If $|z-a| < R$, then the series converges absolutely.

(b) If $|z-a| > R$, then the terms become unbounded and the series diverges.

(c) If $0 < r < R$, then the series converges uniformly on $\{z : |z-a| \leq r\}$,

namely, R is the only number with both (i) and (ii).

Proof. For (a), it suffices to take $a = 0$. Thus if $|z| < R$, we show the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges. Then if $|w-a| < R$, we have that

$$\sum_{n=0}^{\infty} a_n(w-a)^n$$

converges as well. Since $|z| < R$, there exists some $r > 0$ such that $|z| < r < R$. I claim that for this $r \in (0, R)$, there is an $N \in \mathbb{N}$ such that if

$$n \geq N$$

then

$$|a_n|^{\frac{1}{n}} < \frac{1}{r}.$$

(But where did this N come from?) We know by definition that

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}.$$

Since $r < R$, we know $\frac{1}{r} > \frac{1}{R}$ and thus have that

$$\frac{1}{r} - \frac{1}{R} > 0.$$

If we take $\epsilon = \frac{1}{r} - \frac{1}{R}$, we get the existence of $N \in \mathbb{N}$ such that if

$$n \geq N$$

then

$$\left|\sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, ...\} - \frac{1}{R}\right| < \frac{1}{r} - \frac{1}{R}.$$

Thus $\sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, ...\} < \frac{1}{r}$, then by definition of sup,

$$|a_n|^{\frac{1}{n}} < \frac{1}{r}.$$

This completes the proof of my claim. We can now let $n \geq N$, we raise everything to the n, starting with what we obtained from out claim,

$$|a_n| < \frac{1}{r^n}.$$

Thus we have that

$$\begin{aligned} |a_n z^n| &< \frac{|z|^n}{r^n} \\ &= (\frac{|z|}{r})^n \\ &< 1. \end{aligned}$$

Thus we have

$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{N} |a_n z^n| + \sum_{n=N+1}^{\infty} |a_n z^n|$$

Where the first part of the right hand side is finite, and $\sum_{n=N+1}^{\infty} |a_n z^n|$ is dominated by $\sum_{n=N+1}^{\infty} (\frac{|z|}{r})^n$, which is geometric and convergent as $\frac{|z|}{r} < 1$ and therefore $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely as needed. $\square$

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Lecture 3 ; September 4, 2019

From last time we proved pat (a) of the following. big theorem

Theorem 2: (Conway III.1.3) For a given power series centered about a,

$$\sum_{n=0}^{\infty} a_n(z-a)^n,$$

define, for $R \in [0, \infty]$,

$$\frac{1}{R} := \limsup |a_n|^{\frac{1}{n}}.$$

Then,

(a) If $|z-a| < R$, then the series converges absolutely.

(b) If $|z-a| > R$, then the terms become unbounded and the series diverges.

(c) If $0 < r < R$, then the series converges uniformly on $\{z : |z-a| \leq r\}$,

namely, R is the only number with both (i) and (ii).

The proof for (b) is left for the latter portion of this entire proof. We will first prove part (c). In order to do so we utilize the Weierstrass $M-$test which states that for a given sequence of real or complex valued functions, say $\{u_n\}_{n\in\mathbb{N}}$, defined on some set A, and a sequence of non-negative real numbers $\{M_n\}_{n\in\mathbb{N}}$ such that

$$|u_n(x)| \leq M_n$$

for $n \geq 1$ and all $x \in A$. If in addition,

$$\sum_{n=1}^{\infty} M_n$$

converges, then

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely and uniformly on A. This is officially known as the Weierstrass $M-$test.

Proof. For (c), let $r \in (0, R)$. We would like to utilize the Weierstrass $M-$test. Let $M_1, M_2, ...$ be a convergent sequence in $\mathbb{R}$ such that the series converges, say to $M \in \mathbb{R}$. We may assume that for every $n \geq 1$ and all $x \in U \subseteq \mathbb{C}$ we have complex valued functions $u_1(x), u_2(x), ..$ defined on U, such that

$$|u_n(x)| \leq M_n.$$

Then we have that

$$\sum_{n=1}^{\infty} u_n$$

is uniformly convergent on U. Note if there exists an r such that $|z-a|<r$ implies

$$\sum a_n(z-a)^n$$

converges absolutely, then $r \leq R$. but $r \in (0,R)$ thus we are guaranteed the existence of some $\rho \in (r,R)$. By our claim in the proof of part (a), we showed the existence of an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n|^{\frac{1}{n}} < \frac{1}{\rho}.$$

Thus we get that

$$|a_n| < \frac{1}{\rho^n},$$

Then we can write

$$|a_n z^n| < \frac{z^n}{\rho^n}.$$

If $|z|<r$, then we get

$$|a_n z^n| < \left|\frac{r}{\rho}\right|^n$$

By the W-M test, we can take

$$M_n = (\frac{r}{\rho})^n,$$

then we have that

$$\sum_{n=N}^{\infty} a_n z^n$$

converges uniformly on

$$\{z \in \mathbb{C} : |z| \leq r\}.$$

Forcing

$$\sum_{n=1}^{\infty} a_n z^n$$

to converge uniformly on

$$\{z \in \mathbb{C} : |z| \leq r\}.$$

as needed. □

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Lecture 4 ; September 9, 2019

From last time:

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