Notes on Several Complex Variables

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Notes pulled from Complex Geometry by Daniel Huybrechts

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Abstract

In these notes we give a brief overview at the theory of functions of several complex variables. The only prior knowledge to have to make progress in these notes is basic point set topology, introductory Complex Analysis (analytic functions, Cauchy-Riemann equations, Liouvilles Theorem.), Abstract Algebra, and some PDEs. Our goal is to approach the theory first from that of a single complex variable then generalize to higher dimensions whilst trying to not loose track of any complex or algebraic structure.

Contents

Acknowledgements

Before I begin, I would like to start by thanking Daniel Huybrechts and John. B Conway as a vast majority of these notes are pulled from these texts, to avoid any plagarism all credit goes to the two authors. Also I would like to thank the Mathematics deparement at CSULB which is where I did my Masters in Pure Mathematics (2021). Namely, I would like to thank Professors John O. Brevik, Ryan C. Blair, David Gau, and Florence Newberger (in no particular order) as they have matured me mathematically into who I am today. Lastly, I would like to dedicate this to and thank my family. My mother, father, and sister for the continued love and support in my mathematical endeavors all along.

§1.1 Holomorphic functions

The theory of functions of several complex variables is a rich and elegant field; it pulls from various fields such as but not limited to, Topology, Abstract Algebra, and Partial differential equations. One of the most crucial concepts across the board for complex variables is the notion of a *holomorphic* map. Before moving into the land of several complex variables, we must first recall what it means for a function defined over $\mathbb{C}$ to be holomorphic. For the remainder of these notes, $U \in \tau_{\mathbb{C}}$ when defined. That is, U is open in $\mathbb{C}$.

Let $U \subset \mathbb{C}$. We say that a map

$$f : U \to \mathbb{C}$$

is *holomorphic* if for each $z_0 \in U$ there exists an $\epsilon > 0$ such that $f(z)$ converges on $B_\epsilon(z_0)$. Moreover, f is holomorphic if it can be written as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$$

for every $z \in B_\epsilon(z_0)$.

There are many equivalent definitions of holomorphicity over $\mathbb{C}$, however one definition we appeal to most is that of the *Cauchy-Riemann* equations. Note that $z \in \mathbb{C}$ can be written as

$$z = x + iy$$

where $x, y \in \mathbb{R}$. Then f can be regarded as a function $f(x, y)$ of two real variables. As a matter of fact one can write f as

$$f(x, y) = u(x, y) + iv(x, y)$$

where

$$u(x, y) := \mathrm{Re}(f), v(x, y) := \mathrm{Im}(f),$$

the real and imaginary parts respectively. Note that u, v are real-valued functions

$$u, v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Now one can show that f is holomorphic if and only if the Cauchy-Riemann equations are satisfied. That is, if and only if

$$u_x = v_y$$
$$u_y = -v_x.$$

I.e., the derivative of f need be $\mathbb{C}-$linear. This allows us to define differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad (*)$$

and

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \qquad (**)$$

These are motivated by the properties

$$\frac{\partial}{\partial z}(z) = 1 = \frac{\partial}{\partial \overline{z}}(\overline{z}),$$

and

$$\frac{\partial}{\partial z}(\overline{z}) = 0 = \frac{\partial}{\partial \overline{z}}(z),$$

Then the C-R equations can be writtten as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

As the jump from real partial derivatives to complex partial derivatives is vast, we will spend a bit more time on this section. Consider a differentiable map

$$f : U \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2.$$

Then it only makes sense to mention the differential of f at some $z \in U$. Namely, its differential $df(z)$ at some $z \in U$ is the $\mathbb{R}-$linear map

$$df(z) : T_z\mathbb{R}^2 \to T_{f(z)}\mathbb{R}^2$$

between tangent spaces. It is crucial to note the dimension of the tangent space is the same as the dimension of our ambient space. Writing $z = x + iy, w = r + is$ for the two tangent spaces we enjoy a nice canonical bases given via

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle, \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial s} \rangle.$$

With respect to these basis, the differential $df(z)$ is given via the real Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

where $f = u + iv$. I.e., $u = r \circ f$, $v = s \circ f$.

Our goal is to now extend this to a $\mathbb{C}-$linear map, (Recall a map f is $\mathbb{C}-$ linear if

$$f(i) = if(1).)$$

which is given via

$$df(z)_{\mathbb{C}} : T_z\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z)}\mathbb{R}^2 \otimes \mathbb{C}.$$

We can now choose as basis, $(*)$ and $(**)$. With respect to this basis, $df(z)$ is given via

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \\ \frac{\partial \overline{f}}{\partial z} & \frac{\partial \overline{f}}{\partial \overline{z}} \end{pmatrix}.$$

Exercise 1. Show that for any function f,

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\left(\frac{\partial f}{\partial z} \right)}.$$

Exercise 2. If $f = u + iv$ is holomorphic, then

$$\frac{\partial f}{\partial \overline{z}} = 0 = \frac{\partial \overline{f}}{\partial z}.$$

Using results from these two exercises we note that $df(z)$ has a new base given via the diagonal matrix

$$\begin{pmatrix} \frac{\partial f}{\partial z} & 0 \\ 0 & \frac{\partial \overline{f}}{\partial \overline{z}} \end{pmatrix}.$$

Holomorphicity of f is also equivalent to the *Cauchy Integral Formula.* More precisely, a function

$$f : U \to \mathbb{C}$$

is holomorphif if and only if f is continuously differentiable and for any

$$B_\epsilon(z_0) \subset U,$$

we have that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{(z - z_0)} dz.$$

In fact, this formula holds for any function

$$f : \overline{B_\epsilon(z_0)} \to \mathbb{C}$$

which is holomorphic on the interior. Namely, the Cauchy integral formula is used in proving the existence of a (convergent) power series expansion of any function satisfying the Cauchy-Riemann equations. We list a few crucial results from functions of a single variable which will be of use to us.

Maximum Principle Let $U \subset \mathbb{C}$ be open and connected. If

$$f : U \to \mathbb{C}$$

is holomorphic and non-constant, $|f|$ has no local maximum in U. Moreover, if U is bounded and f can be extended to a continuous function

$$\overline{f} : \overline{U} \to \mathbb{C},$$

then $|f|$ takes on its maximal values on ∂U.

Maximum Principle (Alternate) Let $U \subset \mathbb{C}$ is open and connected If

$$f : U \to \mathbb{C}$$

is holomorphic and there exists a point $z_0 \in U$ such that

$$|f(z_0)| \geq |f(z)|$$

for every $z \in U$, then f is constant on U.

Identity Theorem If

$$f, g : U \to \mathbb{C}$$

are holomorphic functions on an open and connected subset $U \subset \mathbb{C}$ such that

$$f(z) = g(z)$$

for each $z \in V \subset U$ some non-empty open subset, then $f = g$.

Riemann extension Theorem Let

$$f : B_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$$

bee a bounded holomorphic function. Then, f can be extended to a holomorphic function

$$f : B_\epsilon(0) \to \mathbb{C}.$$

Riemann Mapping Theorem Let $U \subsetneq \mathbb{C}$ be simply connected. Then $U \cong B_1(0)$. That is, there exists a bijective holomorphic map

$$f : U \to B_1(0)$$

such that f^{-1} is also holomorphic.

In other words, U is conformally equivalent to the open unit disk if U is not all of $\mathbb{C}$ (Tao, 2018).

Liouville If

$$f : \mathbb{C} \to \mathbb{C}$$

is bounded, then f is constant. I.e., Bounded entire functions need be constant. Here entire is used to denote holomorphicity on all of $\mathbb{C}$.

Residue Theorem Let

$$f : B_\epsilon(0) \setminus \{0\} \to \mathbb{C}$$

be a holomorphic map. Then f can be extended via a Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where the coefficient a_{-1} is given by the residue formula

$$a_{-1} = \frac{1}{2\pi i} \int_{|z|=\frac{\epsilon}{2}} f(z) dz.$$

We can now safely extend our notions to functions of several complex variables. The notion can be extended in two ways. Firstly, one could consider a function of several complex variables

$$\mathbb{C}^n \to \mathbb{C}.$$

Secondly, functions taking on values actually in $\mathbb{C}^n$. We must first consider what the correct basis choice would be for higher (complex) dimensions. As a basis for the topology in higher dimensions we will use these *polydisks*

$$B_\epsilon(w) = \{z : |z_j - w_j| < \epsilon_j\},$$

where $\epsilon := \{\epsilon_1, ..., \epsilon_n\}$.

Definition 1.1 Let $U \subset \mathbb{C}$. Let

$$f : U \to \mathbb{C}$$

be continuously differentiable. Then f is said to be *holomorphic* if the Cauchy-Riemann equations hold for all coordinates

$$z_j = x_j + iy_j.$$

I.e.,

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}$$

and

$$\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$$

for $j \in \{1, 2, ..., n\}$.

So by definition a continuously differentiable function f is holomorphic if the induced functions

$$f|_U : U \cap \{(z_1, z_2, ..., z_{j-1}, z, z_{j+1}, ..., z_n) : z \in \mathbb{C}\} \to \mathbb{C}$$

are holomorpic for every choice of j and for fixed $z_1, z_2, ..., z_{j-1}, z, z_{j+1}, ..., z_n \in \mathbb{C}$.

More simply put, a functions holomorphicity depends heavily on its component-wise holomorphicity as j ranges from 1 to n about each point $z \in \mathbb{C}$.

We can now rewrite our differential operators component-wise as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

and

$$\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

By the Cauchy-Riemann equations we have that

$$\frac{\partial f}{\partial \overline{z_j}} = 0$$

for $j \in \{1, 2, ..., n\}$. One tricky observation is that the equations in Definition 1.1 yield

$$\overline{\partial} f = 0.$$

We next turn our attention to the Cauchy integral formula for functions of several variables. Let us first Recall what this means in $\mathbb{C}$:

Cauchy's Integral Formula (for $\mathbb{C}$) Let $U \subset \mathbb{C}$ and

$$f : U \to \mathbb{C}$$

holomorphic. Let γ is a closed curve in U. That is,

$$\gamma : [a, b] \to \mathbb{C}$$

is a continuous map with

$$\gamma(a) = \gamma(b).$$

Suppose $\eta(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus U$, then for $z_0 \in U \setminus \gamma([a, b])$ we have

$$\int_{\gamma} \frac{f(z)}{z - z_0} dz = (2\pi i)\eta(\gamma; z_0) f(z_0)$$

where

$$\eta(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz \in \mathbb{Z},$$

is the *winding number* of γ about z_0. Thus if the winding number about a particular point is 0, the value of the integral is therefore 0 as well.

Cauchys Integral Formula for ($\mathbb{C}^n$) Let

$$f\overline{B_\epsilon(w)} \to \mathbb{C}$$

be a continuous function such that f is holomorphic with respect to each component z_j in any point $z \in B_\epsilon(w)$. Then for any $z \in B_\epsilon(w)$,

$$\int_{|\xi_j - w_j| = \epsilon_j} \frac{f(\xi_1, \xi_2, ..., \xi_n)}{(\xi_1 - z_1)(\xi_2 - z_2)...(\xi_n - z_n)} d\xi_1 d\xi_2 \cdots d\xi_n = (2\pi i)^n f(z).$$

Proof. Repeated application of the Cauchy integral formula for the one variable case allows us to swap the iterated integral for the multiple integral. □

Thus continuous functions on open domains which are holomorphic with respect to each single coordinate (whilst others remain fixed) need be holomorphic! This is more famously known as Osgood's Lemma. The lemma is the specific case of Hartog's Theorem which we will get to momentarily. This result drops the assumption that the function need be continuous). As in the case for $\mathbb{C}$, the above integral can be used in writing out a power series expansion of any holomorphic function $f : U \to \mathbb{C}$. More precisely, for any $w \in U$, there exists a polydisk

$$B_\epsilon(w) \subset U \subset \mathbb{C}^n$$

such that the restriction $f|_{B_\epsilon(w)}$ is given via the power series

$$\sum_{j_1, \ldots, j_n = 0}^{\infty} a_{j_1, j_2, \ldots, j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

with coefficients given via

$$a_{j_1,\dots,j_n} = \frac{1}{j_1!\cdots j_n!}\cdot\frac{\partial^{j_1+\dots+j_n} f}{\partial z_1^{j_1}\cdots \partial z_n^{j_n}}.$$

From our above list of fun facts and theorems, the maximum principle, identity theorem, and Liouville generalize rather nicely into higher dimensions. A particular version of the Riemann Extension Theorem holds true and requires proof. The Riemann mapping theorem fails in higher dimensions however. To see the later, consider the the unit disk in $\mathbb{C}^2$ and the polydisk

$$B_{(1,1)}(0) \subset \mathbb{C}^2$$

which are not biholomorphic to one another. The proof is left as an exercise.

Exercise 3. Show the polydisk $B_{(1,1)}(0)$ and $D := \{z \in \mathbb{C}^2 : ||z|| < 1\}$ are not equivalent.

Often times the holomorphicity of a function of several variables is shown by representing a function as an integral of a known holomorphic function. The following Lemma will be of use to use further down the line.

Lemma 1.2 Let $U \subset \mathbb{C}^n$. Also let $V \subset \mathbb{C}$ be an open neighborhood of the boundary of $B_\epsilon(0) \subset \mathbb{C}$. Assume that

$$f : V \times U \to \mathbb{C}$$

is a holomorphic function. Then

$$g(z) := g(z_1, ..., z_n) := \int_{|\zeta|=\epsilon} f(\xi, z_1, ..., z_n)d\xi$$

is holomorphic on U.

Proof. Let $z \in U$. If $|\xi| = \epsilon$ then there exists a polydisk

$$B_{\delta(\xi)}(\xi) \times B_{\delta'(\xi)}(z) \subset V \times U$$

on which f has a power series expansion. As $\partial B_\epsilon(0)$ is compact we can find a finite number of points, call them

$$\xi_1, ..., \xi_n \in \partial B_\epsilon(0)$$

and positive real numbers

$$\delta(\xi_1), ..., \delta(\xi_n)$$

such that

$$\bigcup(\partial B_\epsilon(0) \cap B_{\frac{\delta(\xi_j)}{2}}(\xi_j))$$

is a disjoint union and

$$\partial B_\epsilon(0) = \bigcup(\partial B_\epsilon(0) \cap \overline{B_{\frac{\delta(\xi_j)}{2}}(\xi_j)}).$$

Hence,

$$g(z) = \int_{|\xi|=\epsilon} f(\xi, z_1, ..., z_n) d\xi$$
$$= \sum_{j=1}^{k} \int_{|\xi|=\epsilon, |\xi_j - \xi| < \frac{\delta(\xi_j)}{2}} f d\xi.$$

Thus each summand is holomorphic since the power series expansion of f converges uniformly on

$$\overline{B_{\frac{\delta(\xi_j)}{2}}(\xi_j)}$$

and thus can be swapped with the integral. □

The next result is a key result in the existence of a holomorphic extension without the assumption the function in question is continuous. Also note this result is only valid in $\mathbb{C}^n$ for $n \geq 2$.

Proposition 1.3: Hartog's Theorem Let $\epsilon = (\epsilon_1, ..., \epsilon_n)$, $\epsilon' = (\epsilon'_1, ..., \epsilon'_n)$be given such that for each $1 \leq j \leq n$ we have

$$\epsilon'_j < \epsilon_j.$$

If $n > 1$, then any holomorphic map

$$f : B_\epsilon(0) \setminus \overline{B_{\epsilon'(0)}} \to \mathbb{C}$$

can uniquely be extended to a holomorphic map

$$f : B_\epsilon(0) \to \mathbb{C}.$$

Proof. It suffices to assume $\epsilon = (1, ..., 1)$. That is, $B_\epsilon(0) \subset \mathbb{C}^n$ is the unit polydisk. As $\epsilon'_j < \epsilon_j$ for each j, there exists $\delta > 0$ such that the open set

$$V := \{z \in \mathbb{C}^n : 1 - \delta < |z_1| < 1, |z_{j \neq 1}| < 1\} \cup \{z \in \mathbb{C}^n : 1 - \delta < |z_2|, |z_{j \neq 2}| < 1\}$$

is properly contained in the complement

$$B_\epsilon(0) \setminus B_{\epsilon'}(0).$$

Note that this gives us an annulus "in between" z_1, z_2. In particular,

$$f : V \to \mathbb{C}$$

is holomorphic. Thus, for any

$$w := (z_2, ..., z_n) \in \mathbb{C}^{n-1},$$

with $|z_j| < 1$, we are guaranteed the existence of a holomorphic function

$$f_w(z_1) := f(z_1, z_2, ..., z_n) \in \mathbb{C}^n$$

on

$$\{z \in \mathbb{C}^n : 1 - \delta < |z_1| < 1, |z_{j \neq 1}| < 1\}.$$

Let

$$f_w(z_1) := \sum_{n=-\infty}^{\infty} a_n(w) z_1^n$$

be the Laurent series of this function. Then the coefficients are given by

$$a_n(w) = \frac{1}{2\pi i} \int_{|\xi| = 1 - \frac{\delta}{2}} \frac{f_w(\xi)}{\xi^{n+1}} d\xi$$

which is holomorphic for w in the unit polydisk of $\mathbb{C}^{n-1}$ by our Lemma. So for a fixed $w \in \mathbb{C}^{n-1}$, the map given via

$$z_1 \mapsto f_w(z_1)$$

is holomorphic on the unit polydisk such that

$$1 - \delta < |z_2| < 1.$$

Thus the negative coefficients are all zero. This forces the coefficients to be identically zero. We can now extend f holomorphically to $\overline{f}$ by the following power series:

$$\sum_{n=0}^{\infty} a_n(w) z_1^n.$$

As the $a_n(w)$ are holomorphic, they attain their max on the boundary. Thus convergence on our annulus yields uniform convergence everywhere. The holomorphic function given in our series glues together with f to give us the desired holomorphism. □

It should be noted that our previous result fails in the one variable case. Informally, Hartog's Theorem says the singularities of f are given by vanishing sets of holomorphic functions.

Next we will examine the Weierstrass preparation theorem (WPT), which will play a key role in the theory of functions of several complex variables.

Let

$$f : B_\epsilon(0) \to \mathbb{C}$$

be holomorphic on the polydisk $B_\epsilon(0)$. For any $w = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$ we let $f_w(z_1)$ denote the function $f(z_1, ..., z_n)$ over $\mathbb{C}^n$. We show that the zeroes of f are caused by a factor of f which has the form of a Weierstrass polynomial.

Definition 1.5 A *Weierstrass polynomial* in z_1 of the form

$$\sum_{j=0}^{d} \alpha_j(w) z_1^{d-j}$$

where the $a_j(w)$ are each holomorphic functions on some small polydisk in $\mathbb{C}^{n-1}$ such that they vanish at the origin.

Recall that any holomorphic function of one complex variable with a zero of order d can be written as

$$z^d h(z),$$

with $h(0) \neq 0$.

Note a zero of order d of $f_0(z_1)$ could decompose into a collection of zeroes of $f_w(z_1)$ whose orders ass up to d.

> **Proposition 1.4: Weierstrass Prep Theorem** Let
>
> $$f : B_\epsilon(0) \to \mathbb{C}$$
>
> be holomorphic on the polydisk $B_\epsilon(0)$. Assume $f(0) = 0$ and $f_0(z_1) \neq 0$. Then there exists a (unique) Weierstrass Polynomial
>
> $$g(z_1, w) = g_w(z_1)$$
>
> and a holomorphic function h on some smaller polydisk $B_{\epsilon'}(0) \subset B_\epsilon(0)$ such that
>
> $$f = g \cdot h$$
>
> and $h(0) \neq 0$.

Proof. Let f be given with multiplicity d and zeroes

$$a_1(w), ..., a_d(w).$$

Since f_0 is not identically zero, we can find an $\epsilon_1 > 0$ such that $f_0 \in \overline{B_{\epsilon_1}(0)}$ vanishes only in 0. Then choose $\epsilon_2, ..., \epsilon_n > 0$ such that

$$f(z_1, z_2, ..., z_n) \neq 0$$

for $|z_1| = \epsilon_1$ and $|z_j| < \epsilon_j$ for $j = 2, ..., n$. Note if $w = 0$ then

$$a_1(0) = ... = a_d(0) = 0.$$

Next, it would be nice if we knew if there was a relationship between d and w. That is, if d depends on w or not. The following polynomial has the same zeroes (with multiplicities) as f_w.

$$g_w(z_1) := \prod_{j=1}^{d} (z_1 - a_j(w)).$$

Thus, for a fixed w, the function

$$h_w(z_1) := \frac{f_w(z_1)}{g_w(z_1)}$$

is holomorphic in z_1. We must now only show $g_w(z_1), h_w(z_1)$ are holomorphic in w. □

(Text not finished yet, work in progress for a while. This is my first attempt at a text book or informal notes on a given topic). *<back2top>*

2. Complex and Hermitian structures

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