GRE Subject: Mathematics
Select Tips and Tricks

MyMathYourMath.com
Typset by Hossien Sahebjame

Spring/Fall 2021

Contents

1 Number Theory

In this section, for any $a, b \in \mathbb{Z}$, let $(a,b), [a,b]$ denote their *gcf* and *lcm* respectively.

- **Divisibility** For any integer $n \ni 1 \leq n \leq a$, the number of n such that $y \mid n$ but $x \nmid n$ is given by $(y > x)$
$$(\frac{a}{y}) - (\frac{a}{[x,y]}).$$
- **Successor existence** For $n \in \mathbb{Z}^+$, if
$$1 \leq x_1 < x_2 < ... < x_{n+1} \leq 2n,$$
then there is an i such that
$$x_{i+1} = x_i + 1.$$
- **Decimal Expansion** Recall: $n!$ has $\sum_{k\in\mathbb{N}} \lfloor \frac{n}{5^k} \rfloor$ zeroes in its decimal expansion
- **Partitions** If n is an integer, number of ordered partitions is 2^{n-1}.
- **Number of divisors** Let $\sigma(n)$ denote the number of divisors of $n \in \mathbb{Z}^+$. If we take $n = \prod_{i=1}^{k} p_i^{q_i}$ as the unique prime factorization of n, then
$$\sigma(n) = \prod_{i=1}^{k} p_i^{q_i - 1},$$
moreover, σ is multiplicative, that is,
$$\sigma(ab) = \sigma(a)\sigma(b).$$
- **Fun Example** If $n = 2^n$ then the sum of divisors is given by
$$\frac{2^{n+1} - 1}{2 - 1}.$$
- **Fun Example** If P is product of any 4 consecutive integers, then $P + 1$ is a perfect square.
- **Fun Example** If $6x + 5y \equiv_{13} 0$, for what b is $8x + by \equiv_{13} 0$:
Find $[8,6] = 24$ to cancel out the x's then you get
$$20y - 3by \equiv_{13} 0$$
in other words
$$7y - 3by \equiv_{13} 0$$

Must find b to satisfy

$$7 - 3b \equiv_{13} 0$$

so $b = 11$.

- **Fun Example** Greatest integer that divides $p^4 - 1$ for prime > 5:
  Write

  $$p^4 - 1 = (p+1)(p-1)(p^2+1).$$

  by the difference of 4ths and squares. Then as $p > 5$, for $(p \pm 1)$, one is divisible by 2, and the other by 4 thus

  $$2 \cdot 4 = 8 \mid (p+1)(p-1).$$

  Moreover $p^2 + 1$ is divisible by 2 thus we have that

  $$16 \mid (p+1)(p-1)(p^2+1).$$

  For any $p \neq 3$ we have that $p \equiv 1 \mod 3$ then we have $p^2 - 1 \equiv 0 \mod 3$ thus

  $$3 \cdot 16 = 48 \mid (p+1)(p-1)(p^2+1)$$

  Lastly, note all odd squares end in 1 or 9 mod 10 thus (p^2+1) or (p^2-1) is divisible by 5 and thus

  $$3 \cdot 5 \cdot 16 \mid p^4 - 1.$$

  is the integer in question.

<back2top>

2 Elementary Algebra/Geometry

- **Min & Max** For all $x, y \in \mathbb{R}$, we have

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}.$$

and

$$\min\{x, y\} = \frac{x + y - |x - y|}{2}.$$

- **Binomial Theorem** For any binomial $(x+y)^n$ we have by the Binomial theorem that

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\end{aligned}$$

Where $\binom{n}{k}$ is the binomial coefficient given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- **Distance between point line** If $y = mx + b$ is a line and (p, q) is a point, then the shortest distance is given by

$$d = \sqrt{\frac{(b + mp - q)^2}{m^2 + 1}}.$$

- **Cubic Discriminant** If $ax^3 + bx^2 + cx + d = 0$ is a cubic, then its discriminant is given by

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

- **Euler Formulas** Recall by Euler we have that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

and

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}.$$

<back2top>

3 Calculus I-III

- **Integrals you should know!**

  1. $\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C.$
  2. $\int \tan x dx = -\ln|\cos x| + C.$
  3. $\int \cos^2 x dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C.$
  4. $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$
  5. $\int \frac{1}{x \log x} dx = \log|\log x| + C.$
  6. $\int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$
  7. $\int e^x \cos x dx = \frac{e^x}{2}(\sin x + \cos x) + C.$
  8. $\int \frac{\cos x}{\cos x + \sin x} dx = \frac{x}{2} + \frac{\ln|\cos x + \sin x|}{2} + C.$
  9. $\int \sin^3 x dx = \frac{\cos^3 x}{3} - \cos x + C.$
  10. $\int \cos^3 x dx = \frac{-\sin^3 x}{3} + \sin x + C.$
  11. $\int_a^b f(x) dx$ can be written as

  $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right)$$

  Where $\Delta x = \frac{(b-a)}{n}$.

- f **VS.** f'

  1. If $f(x)$ is increasing in an interval I, then $f'(x) > 0$ on I.
  2. Similarly, if $f(x)$ is decreasing on an interval I, then $f'(x) < 0$ on I.

- **Tangent Plane** If

  $$z = f(x, y),$$

  then the equation of the plane tangent at (x_0, y_0) is given by

  $$z - f((x_0, y_0)) = z_x((x_0, y_0))(x - x_0) + z_y((x_0, y_0))(y - y_0).$$

- $\frac{d}{dx}$ **of** $|\cdot|$ For the derivative of $|f(x)|$, using the square root of the square we have the derivative as

  $$f'(x)\frac{f(x)}{|f(x)|}.$$

- **Piece-wise Derivative** If f is a function over closed unit interval given as the piece-wise

$$f(x) = \begin{cases} a^2 & x \in [0, a] \\ ax & \text{otherwise} \end{cases}$$

Then to determine the value of $a \in [0, 1]$ such that

$$\int_0^1 f(x)dx = 1,$$

Split up the integral as

$$\int_0^a a^2 dx + \int_a^1 axdx = 1$$

and solve for a.

- **Counter-Examples** Function that is continuous but not uniform

$$f(x) = \frac{1}{x}.$$

Function uniform not Lipschitz

$$f(x) = \sqrt{x}.$$

Function Lipschitz not differentiable is

$$f(x) = |x|.$$

- **FTC** By F.T.C.,

$$\frac{d}{dx} \int_0^{x^2} e^{-t^2} dt = 2xe^{-(x^2)^2}$$

Where the $2x$ gets tagged on by chain rule.

- **Taylor series** For a given function $f(x)$, we can approximate it using the taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

- **Generalized Derivatives**

  1. If $f(x) = e^{kx}x$ then the generalized nth derivative is given via

$$f^{(n)}(x) = (k)^{n-1} e^{kx}(kx + n).$$

2. If $f(x) = x \sin x$ then the nth derivative is given via

$$f^{(n)}(x) = n \sin(x + \frac{(n+1)\pi}{2}) + x \sin(x + \frac{n\pi}{2})$$

3. If $f(x) = \sin x$, then the nth derivative is given by

$$f^n(x) = n \sin(x + \frac{\pi n}{2}).$$

4. If $f(x) = \cos x$, then the nth derivative is given by

$$f^n(x) = n \cos(x + \frac{\pi n}{2}).$$

- **Point & Plane** If $Ax + By + Cz + D = 0$ is a plane and (x_0, y_0, z_0) is a point in 3-space, then the distance is given via

$$d = \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2 + D^2}}.$$

- **Plane & Plane** If $Ax + By + Cz + D_1 = 0, Ax + By + Cz + D_2 = 0$ are parallel planes, the shortest distance between them is given via

$$d = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}.$$

- **Max Value** If $f(x_1, ..., x_n)$ is a function, the max value of the directional derivative at $(a_1, ..., a_n)$ is length of

$$\langle \partial f_{x_1}(a_1, ..., a_n)), ..., \partial f_{x_n}(a_1, ..., a_n)) \rangle$$

- **Curve length** If $x(t), y(t)$ for $t \in [a, b]$ are parameterized curves, then their length is given via the definite integral

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dx,$$

Where $(\frac{dx}{dt}), (\frac{dy}{dt})$ are first partial derivatives of x and y respectively, with respect to t.

- **Fun Example** Note the series

$$\sum_{k=0}^{\infty} \frac{1}{k} - \frac{1}{2^k}$$

goes to ∞.

- **Fun Example** The series $\sum_{k=1}^{\infty}\frac{k^2}{(k-1)!}$ converges to $2e$. To see this note

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{k^2}{(k-1)!} &= \sum_{k=1}^{\infty}\frac{k}{(k-1)!} \\ &= \sum_{k=1}^{\infty}\frac{(k-1)+1}{(k-1)!} \\ &= \sum_{k=2}^{\infty}\frac{1}{(k-2)!}+\sum_{k=1}^{\infty}\frac{1}{(k-1)!} \\ &= e+e \\ &= 2e.\end{aligned}$$

  as needed.

- **Fun Example** If f and g are real-valued and differentiable on closed unit interval and $f'(x) \geq g'(x)$, $\forall x \in [0,1]$, then

$$f(1)-g(1) \geq f(0)-g(0).$$

  This is by Mean value theorem.

<back2top>

4 Analysis/Topology

- **Continuous Piece-Wise** For a piecewise function $g(x)$ with $x \in \mathbb{R}$, defined by 1 for $x \in \mathbb{Q}$ and e^x for $x \in \mathbb{R} \setminus \mathbb{Q}$ continuous for $e^x = 1$ thus only at $x = 0$.
- **LUB Property**If $\emptyset \neq B \subset \mathbb{R}$ has a least upper bound b with $b \notin B$, then b is a limit point of B. (This is because $\overline{B} = B \cup$ limit points of B)
- If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the set

  $$A = \{f(x) : x \in (a, b)\}$$

  is both connected and bounded, in other words images of bounded AND connected are preserved under continuous functions.
- **Metric spaces** If d is a metric on a set S, then

  $$d + k, e^d + k, d^2$$

  are not metrics, however

  $$\sqrt{d}$$

  is.
- **Limit Point Sets** For any topological space X and subset Y, let Y' denote the set of limit points of Y. Then for any subsets A, B of X then

  $$(A \cup B)' = A' \cup B',$$

  and if $A' = \emptyset$ then A is closed.
  However, intersections are not preserved and if $A \in \tau_X$, then A' may or may not be empty.
- **Nested Invariants** If $A \subset B \subset C$, if A,C are connected, compact, or Hausdorff, then B is only guaranteed the Hausdorff property.
- **Lower Limit Topology** Facts about $\mathbb{R}_l$:
  1. Hausdorff
  2. Lindelof
  3. NON COMPACT
  4. TOTALLY DISCONNECTED.
- **K-Topology** Facts about $\mathbb{R}_K$:
  1. Connected but NOT path

2. Hausdorff
3. NON COMPACT

- **Cofinite Topology** Facts about $\mathbb{R}_{\text{cofinite}}$:
  1. NON HAUSDORFF
  2. COMPACT
  3. infinite sets are NOT closed as their complements are NOT open.

- **Mobius Transformation** If $f(z) = \frac{az+b}{cz+d}$ is a given mobius transformation, then the fixed points are given by setting

$$\frac{az+b}{cz+d} = z.$$

  using quadratic to solve for z.

- **Continuity** In the ϵ, δ definition if

$$\lim_{x \to x_0} f(x) = L,$$

  then the least upper bound on δ is given by

$$\frac{\epsilon}{f'(c)}.$$

- **Fun Example** If a metric is defined on Riemann integrable functions

$$d(f, g) = \int_0^1 |f - g| dx,$$

  then

$$d(f, g) > 0$$

  is NOT satisfied if $f \neq g$. Simply define f to be 1 for every $x \in [0, 1]$ and g to be 1 for all the x but equal 2 when $x = 1$.

<back2top>

5 Abstract & Linear Algebra

- **Determinant Properties** For $n \times n$ matrices A, B if $|A|$ denotes determinant of A, then behold, properties of the determinant

  1. $|AB| = |A||B|$
  2. $|cA| = c^n|A|$
  3. $|A^{-1}| = \frac{1}{|A|}$
  4. $|A| = |A^T|$
  5. $|A| = \prod \lambda_i$ where the λ_i are eigen values of A.
  6. Swapping rows negates determinant.
  7. Scalar any row of A by k then new determinant $= k|A|$.
  8. Given a 3×3 matrix

     $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

     We can compute the determinant via

     $$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

  9. (**Relate det & inverse**) For $n = 2$, if A (non-singular) is given via

     $$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

     then its inverse is given via

     $$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- **Dimension of Sum** For two subspaces of dim 10 space,

  $$\dim(V_1 + V_2) \leq 10.$$

  Also

  $$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

  So if V_i are both dim 6, intersection is 2. More generally, for any two vector space or subspace, say W, Z, one has

  $$\dim(W + Z) = \dim W + \dim Z - \dim(W \cap Z).$$

- **Eigen Values** For any given $n \times n$ matrix A, the product of the eigen values is given by

$$\prod_{i=1}^{n} \lambda_i = \det(A - \lambda I).$$

And the sum of the eigen values is given by

$$\sum_{i=0}^{n} \lambda_i = \text{tr}(A).$$

- **Eigen Vectors** If V is vector space of continuous functions and $D : V \to V$ sends $f \to f'$, what are eigen vectors of D?

  We need solutions to $f'(x) = \lambda f(x)$, so all nonzero function of the form

$$ke^{\lambda x}.$$

- **Non - Subring Example** If R is a polynomial ring of polynomials over $\mathbb{R}$. Then the set of even degree polynomials is not a subring as

$$x^2 + x, -x^2$$

  are not closed under addition.

- **Subring Example** If A, B are subspace of vector space V, then $A \cap B$ and $A + B$ are subspaces not complement or union.

- Linear combo of rows does nothing.

- **Iso of** $\mathbb{Q}$ Number of ring isomorphisms of $\mathbb{Q}$ to itself (as a field) is only 1, trivial sending 1 to 1.

- **Similar Matrices** If A, B are similar, then they have same trace, they're inverses are similar, and so are $A - 2I, B - 2I$.

- **IDs & Irreducibles** If D is integral domain of form $r + s\sqrt{17}$ then irreducibles are when

$$(r, s) = 1.$$

- **Einstein Criteria for Irreducibility** If $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. then Einsteins criteria for irreducibility states if there exists a prime p such that

  1. $p | a_j; 0 \leq j < n$.
  2. $p \nmid a_n$

3. $p^2 \nmid a_0$,

Then P is irreducible.

- **Hom/Aut Group**

  1. For the modular groups,

  $$|\text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})| = (m, n),$$

  the *gcd* of m and n.

  2. For a field F with multiplicative and additive map into itself, then

  $$|\text{Aut}(F)| = 1.$$

- **Cauchy #** The *Cauchy Number* for a given element of the symmetric group is the sum of $k_i - 1$ where k_i is the length of cycles.

- **Cycle length is the order** If $\sigma \in S_n$, then the order of σ is his own cycle length.

<back2top>

6 Set Theory & Logic

- If $A = \{1, 2, ..., k\}$ then the number of bijections from A onto itself is $k!$.
- The number of injection from $\{1, 2, ..., m\}$ to $\{1, 2, ..., n\}$ where $m < n$ is equal to
$$(n)(n-1)(n-2)\cdot ... \cdot(n-(m+1)).$$
- Given the statement
$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \ni (f(x) > 0) \Rightarrow (g(y) > 0).$$
Then the negation is given by
$$\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, (f(x) > 0) \wedge (g(y) \leq 0).$$
- Let A, B be subsets of M and $S_0 = \{A, B\}$. If S_{i+1} is defined by taking unions, intersections, and complements, how many elements are in $\bigcup S_i$? 16.
- For any finite set X of n elements,
$$|\mathcal{P}(X)| = 2^n,$$
however in the case that intersection of all is nonempty, the cardinality is
$$2^{n-1}.$$

<back2top>

7 Misc.

- Probability $x \geq yx$ if $x, yz \in [0,1]$ is given via the iterated integral

$$\int_0^1 \int_0^1 (1 - yz) dydz.$$

- Probability of arranging n distinct things is $n!$ so probability of having one formation is $\frac{1}{n!}$.

- If $x = \sqrt{n + \sqrt{n + \sqrt{n + ...}}}$, then $x = \frac{1+\sqrt{4n+1}}{2}$.

- Probability of picking k correct answers out of n question each with x choices is

$$\binom{n}{k} \left(\frac{1}{x}\right)^k \left(\frac{x-1}{x}\right)^{n-k}.$$

- Let X be discrete random variable uniformly distributed with values

$$x_1, ..., x_n.$$

  Then the variance of X is the mean os the squares minus the square of the means.

- Number of ways of choosing m letters from $n > m$ is n^m if you CAN repeat, if you cannot then its

$$(n)(n-1)...(n-k)$$

  where $n - k$ is the mth number down backwards, for example selecting 5 letter code from 7 letters if no repeating is

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot$$

  Where 3 is the mth.

- If $f(x), g(x)$ are solutions to linear homogeneous DE on (a, b) then both 0 and linear combos are solutions, not products.

- *Bernoulli's Equation* is given via

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

  for $n \geq 2$.

- *Ricatti's Equation* is given via

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2,$$

for $n \geq 2$.

<back2top>

8. Conic Sections

- Circle is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

  Where the center is (h, k) and radius is r.

- (Horizontal) Ellipse is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2)}{b^1} = 1,$$

  where

  1. $a < b$.
  2. Major: $(h \pm a, k)$; Minor $(h, k \pm b)$.
  3. Foci: $(h \pm c, k)$ where $c^2 = a^2 - b^2$.

- (Vertical) Ellipse is given by

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2)}{a^1} = 1,$$

  where

  1. $a < b$.
  2. Major: $(h, k \pm a)$; Minor $(h \pm b, k)$.
  3. Foci: $(h, k \pm c)$ where $c^2 = a^2 - b^2$.

- (Transverse horizontal) Hyperbola is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

  where

- (Transverse vertical) Hyperbola is given by

$$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = 1.$$

  where

<back2top>

Even & Odd

- A function, $f(x)$, is said to be *even* if

$$f(-x) = f(x),$$

  for every x in its domain.

- A function, $f(x)$, is said to be *odd* if

$$f(-x) = -f(x),$$

  for every x in its domain.

- Examples of even odd functions.
  1. $f(x) = \sin x$ is odd
  2. $f(x) = \cos x$ is even
  3. $f(x) = \tan x$ is odd
  4. $f(x) = e^x$ is even
  5. $f(x) = x^n$ is even for even n and odd for odd n.

- Rules of even odd functions
  1. (odd)(even)=(odd)
  2. (odd)(odd)=(even)
  3. (even)(even)=(even)
  4. $\frac{\text{(even)}}{\text{(even)}}$=(even)
  5. $\frac{\text{(odd)}}{\text{(even)}}$=(odd)
  6. (even)±(even)=(even)
  7. (odd)±(odd)=(odd)
  8. Sum of even and odd is neither unless one is identically zero.
  9. $\circ$ off odds is odd, $\circ$ of evens is even.

- Note that

$$\int_{-a}^{a} f(x)dx = 0$$

  when $f(x)$ is odd and

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

  when $f(x)$ is even.

<back2top>