Dirichlet's Theorem

Statement of Theorem : If q and l are relatively prime positive integers, then there are infinitely many primes of the form $l + kq$, with $k \in \mathbb{Z}$.

1. **Simple Example**

In this section, we will look at 2 simple examples that will serve as an outline of the proof of Dirichlet's Theorem. The first example is to show that there are infinite primes of the form $6k + 1$ and $6k + 5$. The second example will demonstrate that the specific techniques used in the first example are not sufficient enough to prove there are infinite primes of the form $10k + 1$, and more generalized methods are required.

Example 1. Infinite Primes of the Form $6k + 1$**,** $6k + 5$

The character table for $Z^*(6)$ is as follows.

$\mathbb{Z}^*(6)$	1	5
χ_1	1	1
χ_5	1	-1

We will use the non-trivial character χ_5, and call this χ from here on. We extend χ to all of $\mathbb{Z}$ via:

$$\chi(n) = \begin{cases} 0 \ : n \text{ is even, or divisible by 3.} \\ 1 \ : n \equiv 1 \pmod 6 \\ \text{-1} : n \equiv 5 \pmod 6 \end{cases}$$

Let the L-function $L(s, \chi)$ be defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

then

$$L(1, \chi) = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} ...$$

Since the absolute values of the terms in alternating series $L(1, \chi)$ is monotonically decreasing, by the alternating series test, this series converges to a non-zero finite sum.

We will use the generalized Euler product (proven in section 2),

$$\sum_{n=1} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

as a way to help us control our L-function $L(s, \chi)$. The product above is taken over all primes p.

Taking the logarithm of both sides, we get

$$log(L(s, \chi)) = log(\prod_p \frac{1}{1 - \chi(p)p^{-s}}) = -\sum_p log(1 - \chi(p)p^{-s})$$

Note that $|\chi(p)p^{-s}| < \frac{1}{2}$ since $p \geq 2$ for all p and $|\chi(n)| \leq 1$ for all n.

Combined with the fact that $log(1 + x) = x + O(x^2)$ whenever $|x| < \frac{1}{2}$, we get

$$log(L(s, \chi)) = -\sum_p [-\chi(p)p^{-s} + O(p^{-2s})] = \sum_p \chi(p)p^{-s} + O(1).$$

Since $L(1, \chi)$ is a non-zero finite number, $log(L(s, \chi))$ will be bounded as $s \to 1^+$.

Therefore,

$$\sum_p \frac{\chi(p)}{p^s} = \sum_{p\equiv 1 \pmod 6} \frac{1}{p^s} - \sum_{p\equiv 5 \pmod 6} \frac{1}{p^s}$$

must also be bounded as $s \to 1^+$.

By claim 2 (proven at the end of this section),

$$\sum_{p\equiv 1 \pmod 6} \frac{1}{p^s} + \sum_{p\equiv 5 \pmod 6} \frac{1}{p^s} = \sum_p \frac{1}{p^s}$$

is a diverging series.

From calculus, we know that the sum of a convergent series and a divergent series becomes a divergent series.

Therefore,

$$\sum_p \frac{\chi(p)}{p^s} + \sum_p \frac{1}{p^s} = \sum_{p\equiv 1 \pmod 6} \frac{2}{p^s} = 2 \sum_{p\equiv 1 \pmod 6} \frac{1}{p^s}$$

is a diverging series as $s \to 1^+$.

Hence, $\sum\limits_{p\equiv 1 \pmod 6} \frac{1}{p}$ is a diverging series, and we can conclude that there are infinite primes of the form $6k+1$.

Similarly, since $\sum\limits_p \chi(p)p^{-s}$ is a converging series, multiplying by negative one stlll results in a converging series.

Then

$$\sum_p -\chi(p)p^{-s} + \sum_p \frac{1}{p^s} = \sum_{p\equiv 5 \pmod 6} \frac{2}{p^s} = 2 \sum_{p\equiv 5 \pmod 6} \frac{1}{p^s}$$

is also a diverging series as $s \to 1^+$.

This gives the result that there are infinite primes of the form $6k + 5$.

Note, a key part of this proof lies in the fact that $L(1, \chi) \neq 0$. Since our character χ only took on values of ± 1, we had the alternating series test to help us show $L(1, \chi)$ was converging. What if we were dealing with more complex characters (no pun intended)?

Example 2. Infinite Primes of the Form $10k + 1$

Let's look at the characters of $\mathbb{Z}^*(10)$.

$\mathbb{Z}^*(10)$	1	3	7	9
χ_1	1	1	1	1
χ_3	1	-1	-1	1
χ_7	1	i	$-i$	-1
χ_9	1	$-i$	i	-1

Take χ to be χ_9, and extend it to all of $\mathbb{Z}$ via

$$\chi(n) = \begin{cases} 0 & : n \text{ is even, or divisible by 5.} \\ 1 & : n \equiv 1 \pmod{10} \\ -i & : n \equiv 3 \pmod{10} \\ i & : n \equiv 7 \pmod{10} \\ \text{-1} & : n \equiv 9 \pmod{10} \end{cases}$$

Then

$$L(1,\chi) = 1 - \frac{i}{3} + \frac{i}{7} - \frac{1}{9} + \frac{1}{11} - \frac{i}{13} + \frac{i}{17} - \frac{1}{19} + ...$$

We can separate $L(1,\chi)$ into $(1 - \frac{1}{9} + \frac{1}{11} - \frac{1}{19} + ...) + i(\frac{-1}{3} + \frac{1}{7} - \frac{1}{13} + \frac{1}{17} - ...)$

By the alternating series test, both $(1 - \frac{1}{9} + \frac{1}{11} - \frac{1}{19} + ...)$ and $i(\frac{-1}{3} + \frac{1}{7} - \frac{1}{13} + \frac{1}{17} - ...)$ are converging series that converge to non-zero values.

We can continue with a similar argument used in the Example 1, but we run into problems when adding $\sum_p \frac{1}{p}$ to $\sum_p \chi(p)p^{-s}$ as the imaginary terms do not cancel out.

Therefore, we need more generalized techniques if we are going to prove Dirichlet's Theorem.

■Claims Used In Section

We will prove claim 1, which is then used to help prove claim 2.

Claim 1 (Theorem 1.10 Pg. 249) :

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \text{ when } s > 1$$

Proof of Claim:

We know the sum on the left hand side is the zeta function $\zeta(s)$, which converges for $s > 1$.

We will expand the infinite product on the right hand side, with a relabeling of the primes p into $p_1, p_2, p_3...$ in increasing order.

$$\prod_p \frac{1}{1-p^{-s}} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-s}} = \prod_{i=1}^{\infty} (1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} ...)$$

$$\prod_{i=1}^{\infty} (1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} ...) = 1 + \sum_{1 \le i} \frac{1}{p_i^s} + \sum_{1 \le i \le j} \frac{1}{(p_i p_j)^s} + \sum_{1 \le i \le j \le k} \frac{1}{(p_i p_j p_k)^s} + ...$$

By the fundamental theorem of Arithmetic, each $(p_i p_j p_k ...)$ results in a unique integer n.

So the sum of summations is equal to $\sum_{n=1}^{\infty} \frac{1}{n^s}$

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Claim 2 (Proposition 1.11 Pg. 251) :

$$\sum_p \frac{1}{p} \text{ diverges,}$$

where the sum is taken over all primes p.

Proof of Claim:

By claim 1,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \text{ when } s > 1$$

Taking logarithms of $\zeta(s)$ and the infinite product, and using the fact that $log(1+x) = x + O(x^2)$ when $|x| < \frac{1}{2}$, we get

$$log(\zeta(s)) = log(\prod_p \frac{1}{1-p^{-s}}) = -\sum_p log(1-p^{-s}) = -\sum_p (-p^{-s} + O(p^{-2s})) = \sum_p \frac{1}{p^s} + O(1)$$

We know from calculus that $\sum_{n=1}^{\infty} \frac{1}{n}$ is a diverging harmonic series.

Therefore, $log(\zeta(s)) \to \infty$ as $s \to 1^+$.

Since $O(1)$ is just some constant finite number, we can conclude that $\sum_p \frac{1}{p^s} \to \infty$ as $s \to 1^+$.

Hence, $\sum_p \frac{1}{p}$ diverges.

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2. **Generalized Euler Product**

In this section, we will prove the equality below, which was used in example 1 of section 1.

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

Before we start showing these 2 equations are equal, we must first establish that both equations are converging. Since the techniques used to show equality assumes this fact.

Since

$$|\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}| \leq |\sum_{n=1}^{\infty} \frac{1}{n^s}|$$

and $|\sum_{n=1}^{\infty} \frac{1}{n^s}|$ is a converging series when $s > 1$, we can conlude that $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is an absolutely converging series.

By claim 4 (proven at the end of this section),

$$\prod_p \frac{1}{1-\chi(p)p^{-s}} \text{ converges}$$

Now that we have established that the equations on both sides of the equals sign of the generalized euler product converges, we can now prove the equality.

Let

$$L = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, P = \prod_p \frac{1}{1-\chi(p)p^{-s}}, \text{ the goal is to show that } L = P$$

Define

$$P_N = \prod_{p \leq N} \frac{1}{1-\chi(p)p^{-s}}$$

for some $N \in \mathbb{Z}^+$.

Note that trivially, $P_N \to P$ as $N \to \infty$

We will index primes p in the finite product P_N with p_i, where $p_i < p_{i+1}$, and p_M is the largest prime less than or equal to N.

$$P_N = \prod_{i=1}^{M} \frac{1}{1-\chi(p_i)p_i^{-s}} = \prod_{i=1}^{M} (1 + \frac{\chi(p_i)}{p_i^s} + \frac{\chi(p_i^2)}{p_i^{2s}} + ...)$$

Since χ is multiplicative, the above finite product simplifies to

$$1 + \sum_{1 \leq i}^{M} \frac{\chi(p_i)}{p_i^s} + \sum_{1 \leq i \leq j}^{M} \frac{\chi(p_i p_j)}{(p_i p_j)^s} + \sum_{1 \leq i \leq j \leq k}^{M} \frac{\chi(p_i p_j p_k)}{(p_i p_j p_k)^s} + ...$$

By the fundamental theorem of Arithmetic, each $(p_i p_j p_k ...)$ results in a unique integer n.

Then this becomes the sum

$$\sum_{n \in A} \frac{\chi(n)}{n^s},$$

where A is the set of all integers n, where all its prime factors are less than or equal to N.

Let B be the set of all integers n which has a prime factor greater than N, then

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n \in A} \frac{\chi(n)}{n^s} + \sum_{n \in B} \frac{\chi(n)}{n^s}$$

Combined with the result above that

$$P_N = \prod_{i=1}^{M} \frac{1}{1 - \chi(p_i) p_i^{-s}} = \sum_{n \in A} \frac{\chi(n)}{n^s},$$

we get

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} - P_N = \sum_{n \in B} \frac{\chi(n)}{n^s}.$$

By the triangle inequality,

$$|\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} - P_N| \leq \sum_{n \in B} |\frac{\chi(n)}{n^s}| \leq \sum_{n > N} |\frac{\chi(n)}{n^s}|.$$

As $N \to \infty$, $\sum_{n>N} |\frac{\chi(n)}{n^s}| \to 0$, therefore we can conlude that

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p} \frac{1}{1 - \chi(p) p^{-s}}$$

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■Claims Used In Section

We start by definining

$$\log_1 \frac{1}{1-z} = \sum_{k=1}^{\infty} \frac{z^k}{k} \text{ for } |z| < 1$$

We will prove properties of this log_1 function in claim 3, and use these properties to prove claim 4, which was used in this section.

Claim 3 (Proposition 3.1 Pg. 258) : $\log_1$ satisfies the following properties:
(i). If $|z| < 1$, then

$$e^{\log_1(\frac{1}{1-z})} = \frac{1}{1-z}$$

(ii). If $|z| < 1$, then

$$\log_1 \frac{1}{1-z} = z + E_1(z)$$

where the error E_1 satisfies $|E_1(z)| \leq |z|^2$ whenever $|z| < \frac{1}{2}$.

(iii). If $|z| < \frac{1}{2}$, then

$$|\log_1(\frac{1}{1-z})| \leq 2|z|$$

• *Proof of (i).*

Let $|z| < 1$.

Using complex notation, $z = re^{i\theta}$, where $0 \leq r < 1$.
Note, to prove the (i). it suffices to show that

$$(1-z)e^{\log_1(\frac{1}{1-z})} = 1.$$

After substituting $re^{i\theta}$ for z, we get

$$(1-re^{i\theta})e^{\log_1(\frac{1}{1-re^{i\theta}})} = 1$$

From how $\log_1$ was defined, this expands to

$$(1-re^{i\theta})e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}} = 1$$

So to prove (i). it suffices to prove this last equality above.

Note, if we set $r = 0$, the equality follows. So the plan is to take the derivative of the left hand side with respect to r and show that it is 0. Therefore, the left hand side is constant with respect to r, and we are done.

The derivative with respect to r is

$$(e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}})(\sum_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k})' - (e^{i\theta})(e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}}) - (re^{i\theta})(e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}})(\sum_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k})'$$

This simplifies to

$$[(1-re^{i\theta})(\sum_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k})' - e^{i\theta}](e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}})$$

The derivativive of the summation portion simplifies to

$$(\sum_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k})' = \sum_{k=1}^{\infty}(re^{i\theta})^{k-1}e^{i\theta} = e^{i\theta}\sum_{k=1}^{\infty}(re^{i\theta})^{k-1}$$

$\sum\limits_{k=1}^{\infty}(re^{i\theta})^{k-1}$ is a converging geometric series since $|re^{i\theta}| = |z| < 1$, so it is equal to $\frac{1}{1-re^{i\theta}}$.

Then

$$[(1-re^{i\theta})(\sum_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k})' - e^{i\theta}](e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}}) = [(1-re^{i\theta})\frac{e^{i\theta}}{1-re^{i\theta}} - e^{i\theta}](e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}}) = 0$$

So we have shown that the derivative with respect to r is 0, therefore

$$(1-re^{i\theta})e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}} \text{ is constant for all values of } r, \text{ so } (1-re^{i\theta})e^{\sum\limits_{k=1}^{\infty}\frac{(re^{i\theta})^k}{k}} = 1$$

•*Proof of (ii).*

Let $|z| < 1$.

First, define $E_1(z)$ as follows

$$E_1(z) = \log_1(\frac{1}{1-z}) - z = (z + \frac{z^2}{2} + \frac{z^3}{3} + ...) - z = \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + ...$$

Using the triangle inequality,

$$|E_1(z)| = |\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + ...| \leq |\frac{z^2}{2}| + |\frac{z^3}{3}| + |\frac{z^4}{4}| + ... \leq |\frac{z^2}{2}| + |\frac{z^3}{2}| + |\frac{z^4}{2}| + ... = |\frac{z^2}{2}|(1 + |z| + |z^2| + |z^3| + ...).$$

Since $|z| < \frac{1}{2}$ by hypothesis, $(1 + |z| + |z^2| + |z^3| + ...) \leq (1 + |\frac{1}{2}| + |\frac{1}{4}| + |\frac{1}{8}| + ...) = 2$.

So $|E_1(z)| \leq |\frac{z^2}{2}| \times 2 = |z|^2$.

•*Proof of (iii).*

Let $|z| < \frac{1}{2}$.

Note, if $|z| = 0$, then trivially, $|\log_1(\frac{1}{1-z})| \leq 2|z|$ holds.

So suppose $|z| \neq 0$,

$$|\frac{\log_1(\frac{1}{1-z})}{z}| = |\frac{z + E_1(z)}{z}| \leq 1 + |\frac{E_1(z)}{z}| \leq 1 + |\frac{z^2}{z}| \leq 1 + |z|$$

Then

$$|\log_1(\frac{1}{1-z})| \leq |z| + |z|^2.$$

Since $|z| < \frac{1}{2}$, $|z^2| \leq |z|$.

So $|\log_1(\frac{1}{1-z})| \leq 2|z|$.

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Claim 4 (Proposition 3.2 Pg. 259) : If $\sum |a_n|$ converges with $a_n \neq 1$ for all n, then $\prod_{n=1}^{\infty} \frac{1}{1-a_n}$ converges. Moreover, this product is non-zero.

Proof of Claim :

Let $\sum |a_n|$ be a converging series with $a_n \neq 1$ for all n. Then there exists a $N > 0$ such that for all $n \geq N$, $|a_n| \leq \frac{1}{2}$.

We split our infinite product into 2 parts,

$$\prod_{n=1}^{\infty} \frac{1}{1-a_n} = (\prod_{n=1}^{N} \frac{1}{1-a_n})(\prod_{n=N+1}^{\infty} \frac{1}{1-a_n})$$

Trivially, the finite product $\prod_{n=1}^{N} \frac{1}{1-a_n}$ converges, and is non-zero. So we just need to prove that $\prod_{n=N+1}^{\infty} \frac{1}{1-a_n}$ also converges and is non-zero.

Since $|a_n| < \frac{1}{2}$, by claim 3 (i),

$$\prod_{n=N+1}^{K} \frac{1}{1-a_n} = \prod_{n=N+1}^{K} e^{\log_1(\frac{1}{1-a_n})} = e^{\sum_{N+1}^{K} \log_1(\frac{1}{1-a_n})}.$$

Since $|a_n| < \frac{1}{2}$, by claim 3 (iii),

$$|\sum_{N+1}^{K} \log_1(\frac{1}{1-a_n})| \leq 2\sum_{N+1}^{K} |a_n|.$$

Since $\sum |a_n|$ converges, there exists some finite A such that $\lim_{K\to\infty} \sum_{N+1}^{K} \log_1(\frac{1}{1-a_n}) = A$.

Therefore, $\lim_{K\to\infty} \prod_{n=N+1}^{K} \frac{1}{1-a_n} = e^A$, and since A is finite, e^A is finite and non-zero.

Since both $\prod_{n=1}^{N} \frac{1}{1-a_n}$ and $\prod_{n=N+1}^{\infty} \frac{1}{1-a_n}$ are finite and non-zero, $\prod_{n=1}^{\infty} \frac{1}{1-a_n}$ is finite and non-zero.

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3. L **- Function Properties 1.**

In this section, we will prove four properties of the L-function. Claim 6 will be used to prove claim 7, and claim 7 will be used to prove claim 8. In section 2, we defined the log_1 function, in this section we also define a log_2 function.

$$log_2(L(s,\chi)) = -\int_s^\infty \frac{L'(t,\chi)}{L(t,\chi)}, \text{ where } s > 1, \text{ and } \chi \text{ is a non-trivial character}$$

Recall from section 1 that

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \text{ when } s > 1$$

Claim 5 (Proposition 3.3 Pg. 261) : Fix an positive integer q. Suppose χ_0 is the trivial Dirichlet character of $\mathbb{Z}^*(q)$

$$\chi_0(n) = \begin{cases} 1 : n, q \text{ are relatively prime} \\ 0 : \text{otherwise} \end{cases}$$

and $q = p_1^{a_1} p_2^{a_2} ... p_N^{a_N}$ is the prime factorization of q, then

$$L(s,\chi_0) = (1-p_1^{-s})(1-p_2^{-s})...(1-p_N^{-s})\zeta(s)$$

Proof of Claim :

Let χ_0 be the trivial Dirichlet character of $\mathbb{Z}^*(q)$, where q is a positive integer with prime factorization $p_1^{a_1} p_2^{a_2} ... p_N^{a_N}$.

Then $\chi_0(p_i) = 0$ for $i \in [1, N]$.

Using the generalized Euler product identity,

$$L(s,\chi_0) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \prod_p \frac{1}{1-\chi_0(p)p^{-s}} = \prod_{p\in A} \frac{1}{1-p^{-s}},$$

where the set A contains all prime numbers except $p_1, p_2, ..., p_N$.

Let set B be the set containing only the prime numbers $p_1, p_2, ..., p_N$.

Then

$$L(s,\chi_0) \prod_{p\in B} \frac{1}{1-p^{-s}} = \prod_p \frac{1}{1-p^{-s}} = \zeta(s)$$

Therefore,

$$L(s,\chi_0) = (\prod_{p\in B} 1-p^{-s})\zeta(s) = (1-p_1^{-s})(1-p_2^{-s})...(1-p_N^{-s})\zeta(s)$$

■

Claim 6 (Lemma 3.5 Pg. 262) : Fix an integer q. Suppose χ is a non-trivial Dirichlet character of of $\mathbb{Z}^*(q)$, then

$$|\sum_{n=1}^{k} \chi(n)| \leq q \text{ for any } k$$

Proof of Claim :

Let k be a positive integer and χ be a non-trivial Dirichlet character of $\mathbb{Z}^*(q)$, where q is a positive integer.

Let S denote the sum

$$\sum_{n=1}^{q} \chi(n).$$

Choose any $a \in \mathbb{Z}^*(q)$.
Because of the multiplicative property of χ,

$$\chi(a)S = \chi(a)\sum_{n=1}^{q} \chi(n) = \sum_{n=1}^{q} \chi(an) = \sum_{n=1}^{q} \chi(n) = S.$$

Since a was chosen arbitrarily, $\chi(a)S = S$ for all $a \in \mathbb{Z}^*(q)$, and there exists an $a \in \mathbb{Z}^*(q)$ such that $\chi(a) \neq 0, 1$ because χ is a non-trivial character, we can conclude that $S = 0$.

By the Euclidean Algorithm, $k = pq + r$, for integers p, r with $0 \leq r < q$.
Then

$$\sum_{n=1}^{k} \chi(n) = \sum_{n=1}^{pq+r} \chi(n) = \sum_{n=1}^{pq} \chi(n) + \sum_{n=pq+1}^{pq+r} \chi(n),$$

Since the values of χ repeat and $\sum_{n=1}^{q} \chi(n) = 0$, $\sum_{n=1}^{pq} \chi(n) = 0$.

So

$$\sum_{n=1}^{k} \chi(n) = \sum_{n=pq+1}^{pq+r} \chi(n).$$

Since there are no more than q terms in the sum $\sum_{n=pq+1}^{pq+r} \chi(n)$, and $|\chi(n)| \leq 1$, we can conlude that

$$|\sum_{n=1}^{k} \chi(n)| \leq q.$$

■

Claim 7 (Lemma 3.4 Pg. 261) : Fix an integer q. Suppose χ is a non-trivial Dirichlet character of of $\mathbb{Z}^*(q)$, then

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

converges for $s > 0$, and denote it's sum as $L(s, \chi)$. Also:
(i). The function $L(s, \chi)$ is continuously differentiable for $0 < s < \infty$
(ii). There exists constants $c, d > 0$ such that

$$L(s, \chi) = 1 + O(e^{-cs}) \text{ as } s \to \infty$$

$$L'(s, \chi) = O(e^{-ds}) \text{ as } s \to \infty$$

Proof of Claim :

• *Proof of (i).*

Let $s > 0$ and χ be a non-trivial Dirichlet character of $\mathbb{Z}^*(q)$, where q is a positive integer.

Let $s_k = \sum_{n=1}^{k} \chi(n)$, where $s_0 = 0$, then $\chi(k) = s_k - s_{k-1}$.

Let N be some arbitrary positive integer.

$$\sum_{k=1}^{N} \frac{\chi(k)}{k^s} = \sum_{k=1}^{N} \frac{s_k - s_{k-1}}{k^s} = \frac{s_1 - s_0}{1^s} + \frac{s_2 - s_1}{2^s} + ... + \frac{s_N - s_{N-1}}{N^s}$$
$$= \frac{s_1}{1^s} + \frac{0}{1^s} + \frac{s_2}{2^s} - \frac{s_1}{2^s} + \frac{s_3}{3^s} - \frac{s_2}{3^s} + ... + \frac{s_N}{N^s} - \frac{s_{N-1}}{N^s}$$
$$= \frac{s_N}{N^s} + \sum_{k=1}^{N-1} s_k\left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right)$$

Lets first look at the term

$$\frac{s_N}{N^s} = \frac{1}{N^s} \sum_{k=1}^{N} \chi(n).$$

By claim 6,

$$\left|\frac{1}{N^s} \sum_{k=1}^{N} \chi(n)\right| \leq \left|\frac{q}{N^s}\right|, \text{ a finite number.}$$

Next, lets look at

$$\sum_{k=1}^{N-1} s_k\left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right).$$

Define

$$g(x) = \frac{1}{x^s}, \text{ then } g'(x) = \frac{-s}{x^{s+1}}.$$

By the mean value theorem, there exists a $c \in [k, k+1]$ such that

$$g'(c) = \frac{-s}{c^{s+1}} = \frac{1}{k+1} - \frac{1}{k},$$

after multiplying by a negative 1 on both sides we get

$$\frac{s}{c^{s+1}} = \frac{1}{k} - \frac{1}{k+1}.$$

Since $c \geq k$, this gives us the inequality

$$\frac{s}{k^{s+1}} \geq \frac{1}{k} - \frac{1}{k+1}.$$

Therefore by claim 6,

$$\left|s_k\left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right)\right| \leq \frac{qs}{k^{s+1}},$$

and

$$\left|\sum_{k=1}^{N-1} s_k\left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right)\right| \leq \sum_{k=1}^{N-1} \left|\frac{qs}{k^{s+1}}\right| = qs \sum_{k=1}^{N-1} \frac{1}{k^{s+1}}.$$

Since $s > 0$,

$$\sum_{k=1}^{N-1} \frac{1}{k^{s+1}} \text{ is a converging harmonic series.}$$

So in summary, we found that

$$|\sum_{k=1}^{N} \frac{\chi(k)}{n^s}| \leq |\frac{q}{N^s}| + qs \sum_{k=1}^{N-1} \frac{1}{k^{s+1}},$$

therefore $\sum\limits_{k=1}^{\infty} \frac{\chi(k)}{k^s}$ converges absolutely and uniformly for $s > 0$. This implies that $L(s, \chi)$ is continuous for $s > 0$.

Next, we need to show that $L(s, \chi)$ is also differentiable. To do so, we will take the derivative of the infinite series $\sum\limits_{k=1}^{\infty} \frac{\chi(k)}{k^s}$ and show that it is converging. This is done in a similar method as above.

The derivative is

$$\sum_{k=1}^{\infty} \frac{ln(k)\chi(k)}{k^s}.$$

Let N be some arbitrary positive integer.

$$\sum_{k=1}^{N} \frac{ln(k)\chi(k)}{k^s} = \sum_{k=1}^{N} \frac{ln(k)(s_k - s_{k-1})}{k^s} = \frac{ln(1)(s_1 - s_0)}{1^s} + \frac{ln(2)(s_2 - s_1)}{2^s} + ... + \frac{ln(N)(s_N - s_{N-1})}{N^s}$$

$$= \frac{ln(2)s_2}{2^s} - \frac{ln(2)s_1}{2^s} + \frac{ln(3)s_3}{3^s} - \frac{ln(3)s_2}{3^s} + ... + \frac{ln(N)s_N}{N^s} - \frac{ln(N)s_{N-1}}{N^s}$$

$$= -\frac{ln(2)s_1}{2^s} + \frac{ln(N)s_N}{N^s} + \sum_{k=2}^{N-1} (\frac{ln(k)s_k}{k^s} - \frac{ln(k+1)s_k}{k+1^s}).$$

The sum of the first two terms of the sum above is finite as $N \to \infty$.

Let

$$g(x) = \frac{ln(x)}{x^s}, \text{ then } g'(x) = \frac{1 - sln(x)}{x^{s+1}}$$

By the mean value theorem, there exists a $c \in [k, k+1]$ such that

$$\frac{1 - sln(c)}{c^{s+1}} = \frac{ln(k+1)}{k+1^s} - \frac{ln(k)}{k^s},$$

after multiplying by a negative 1 on both sides we get

$$\frac{sln(c) - 1}{c^{s+1}} = \frac{ln(k)}{k^s} - \frac{ln(k+1)}{k+1^s}.$$

Since $c \geq k \geq 2$, for sufficiently large c, k,

$$\frac{ln(c)}{c^s} \leq \frac{ln(k)}{k^s}$$

and

$$\frac{sln(c) - 1}{c^{s+1}} \leq \frac{sln(k) - 1}{k^{s+1}} \leq \frac{sln(k)}{k^{s+1}}$$

Next, we will use the integral test to determine that

$$\sum_{k=2}^{\infty} \frac{sln(k)}{k^{s+1}} \text{ is convergent.}$$

$$\int_2^{\infty} \frac{sln(x)}{x^{s+1}} dx = \frac{sln(2)+1}{s2^s} = \frac{ln(2)+\frac{1}{s}}{2^s} \text{ which is defined and finite for } s > 0.$$

Therefore the derivative is a converging series and we can conclude that $L(s, \chi)$ is also differentiable.

Hence, $L(s, \chi)$ is continuously differentiable for $s > 0$.

- *Proof of (ii).*

Suppose that s is very large, since we are supposing this statement as $s \to \infty$.

$$|L(s,\chi) - 1| = |\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} - 1| = |\sum_{n=2}^{\infty} \frac{\chi(n)}{n^s}| \leq |\sum_{n=2}^{\infty} \frac{1}{n^s}| \leq |\frac{O(1)}{2^s}|.$$

Then

$$L(s,\chi) = 1 + \frac{O(1)}{2^s} = 1 + O(2^{-s}) = 1 + O(e^{-ln(2)s}) \text{ as } s \to \infty.$$

Set $c = -ln(2)$, and we see that $L(s, \chi) = 1 + O(e^{-cs})$ as $s \to \infty$.

Again, suppose that s is very large.

$$|L'(s,\chi)| = |\sum_{n=1}^{\infty} \frac{ln(n)\chi(n)}{n^s}| \leq |\sum_{n=2}^{\infty} \frac{ln(n)}{n^s}| \leq |\sum_{n=2}^{\infty} \frac{1}{n^{s-1}}| \leq |\frac{O(1)}{2^{s-1}}|.$$

Therefore

$$L'(s,\chi) = \frac{O(1)}{2^{s-1}} = O(2^{-s+1}) = O(e^{-ln(2)(s-1)}) = O(e^{-ln(2)(s)}) \text{ as } s \to \infty.$$

Set $d = -ln(2)$, and we see that $L(s, \chi) = O(e^{-ds})$ as $s \to \infty$.

■

$$log_2(L(s,\chi)) = -\int_s^{\infty} \frac{L'(t,\chi)}{L(t,\chi)}, \text{ where } s > 1, \text{ and } \chi \text{ is a non-trivial character}$$

Claim 8 (Proposition 3.6 Pg. 264) : If $s > 1$, then

$$e^{log_2(L(s,\chi))} = L(s,\chi), \text{ and } log_2(L(s,\chi)) = \sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})$$

Proof of Claim :

Let $s > 1$.

To show the first equality, it suffices to show that

$$e^{-log_2(L(s,\chi))} L(s,\chi) = 1$$

Note that

$$e^{-log_2(L(s,\chi))} L(s,\chi) = e^{\int_s^{\infty} \frac{L'(t,\chi)}{L(t,\chi)}} L(s,\chi),$$

and as $s \to \infty$

$$e^{\int_s^\infty \frac{L'(t,\chi)}{L(t,\chi)}} L(s,\chi) \to e^0 L(s,\chi) = 1$$

Since $e^{-log_2(L(s,\chi))}L(s,\chi) = 1$ as $s \to \infty$, if we can show that $e^{-log_2(L(s,\chi))}L(s,\chi)$ is a constant for all s, then we are done.

Taking the derivative of $e^{-log_2(L(s,\chi))}L(s,\chi)$ with respect to s, we get

$$-\frac{L'(s,\chi)}{L(s,\chi)} e^{-log_2(L(s,\chi))} L(s,\chi) + e^{-log_2(L(s,\chi))} L(s,\chi) L'(s,\chi) = 0$$

Therefore, $e^{-log_2(L(s,\chi))}L(s,\chi)$ is a constant for all s.

To show the second equality, first we fix s and take the exponent of both sides.

The left side becomes

$$e^{log_2(L(s,\chi))} = L(s,\chi)) \text{ by the equality we just shown}$$

The right side becomes

$$e^{\sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})} = \prod_p e^{log_1(\frac{1}{1-\chi(p)p^{-s}})} = \prod \frac{1}{1-\chi(p)p^{-s}} = L(s,\chi)$$

by claim 3 and generalized Euler product.

Since by taking exponents of both sides, they are equal, the difference in $log_2(L(s,\chi))$ and $\sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})$ must be a $2\pi i M(s)$, where $M(s)$ is some integer number dependent on the value of s.

Both $log_2(L(s,\chi))$ and $\sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})$ are continuous functions of s, therefore $M(s)$ must also be a continuous function of s. However, since $M(s)$ is a integer number and continuous function of s, $M(s)$ must then be a constant number.
As $s \to \infty$

$$log_2(L(s,\chi)) = -\int_s^\infty \frac{L'(t,\chi)}{L(t,\chi)} \to 0$$

$$\sum_p log_1(\frac{1}{1-\chi(p)p^{-s}}) = \sum_p \sum_{k=1}^\infty \frac{(\chi(p)p^{-s})^k}{k} \to 0$$

Therefore, $M(s) = 0$, and we can conlude that

$$log_2(L(s,\chi)) = \sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})$$

■

4. **Fourier Analysis, Dirichlet Characters, and Reduction of the Theorem**

In this section, we will combine some of the major claims and conclusions we have and see what else is needed in order to prove Dirichlets Theorem.

We start with our given, two relatively prime positive integers q, l, and our goal is to show there are infinitely many primes of the form $l + kq$, where $k \in \mathbb{Z}$.

As was demonstrated in example 1 of section 1, we will take a non-trivial character χ of $\mathbb{Z}^*(q)$, and extend it to all $\mathbb{Z}$.

Next, we define $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.

By the Generalized Euler Product proven in section 2, and claim 8 proven in the previous section,

$$log_2(L(s, \chi)) = \sum_p log_1(\frac{1}{1 - \chi(p)p^{-s}}).$$

By claim 3 (ii).

$$\sum_p log_1(\frac{1}{1 - \chi(p)p^{-s}}) = \sum_p (\frac{\chi(p)}{p^s} + O(\frac{\chi(p)}{p^s})) = O(1) + \sum_p \frac{\chi(p)}{p^s}$$

$\star$*Suppose without proof that* $L(1, \chi) \neq 0$.

Since $L(s, \chi)$ is a continuously differentiable function of s by claim 7, then $L(s, \chi) \not\to 0$ as $s \to 1^+$. Hence, $log_2(s, \chi)$ will be bounded as $s \to 1^+$. This gives us the result that $\sum_p \frac{\chi(p)}{p^s}$ is bounded as $s \to 1$.

Now, how does $\sum_p \frac{\chi(p)}{p^s}$ being bounded help us with the proof of Dirichlet's Theorem?

We will prove a useful identity using the same given values of Dirichlet's Theorem.

Let q, l be two relatively prime positive integers.
With this q, we can create the group $\mathbb{Z}^*(q)$.
From how $\mathbb{Z}^*(q)$ is defined, $|\mathbb{Z}^*(q)| = \phi(q)$, where ϕ is the Euler Phi function.

We define the function δ_l on $\mathbb{Z}^*(q)$ via

$$\delta_l(n) = \begin{cases} 1: \text{if } n \equiv l \bmod q \\ 0: \text{ otherwise.} \end{cases} \text{, where } n \in \mathbb{Z}^*(q)$$

We can expand δ_l in a Fourier series as follows:

$$\delta_l(n) = \sum_{e \in \widehat{\mathbb{Z}^*(q)}} \widehat{\delta_l}(e)e(n)$$

where

$$\widehat{\delta_l}(e) = \frac{1}{\phi(q)} \sum_{a \in \mathbb{Z}^*(q)} \delta_l(a)\overline{e(a)}$$

Since q, l are relatively prime, there can only be one element m in $\mathbb{Z}^*(q)$ such that $m \equiv l \bmod q$.
For all other elements, their δ_l value will be 0.
Hence,

$$\widehat{\delta_l}(e) = \frac{1}{\phi(q)} \sum_{a \in \mathbb{Z}^*(q)} \delta_l(a)\overline{e(a)} = \frac{1}{\phi(q)}\overline{e(l)}$$

This simplifies the δ_l Fourier series into

$$\delta_l(n) = \frac{1}{\phi(q)} \sum_{e \in \widehat{\mathbb{Z}^*(q)}} \widehat{e(l)} e(n)$$

We can extend the function δ_l to all of $\mathbb{Z}$ via

$$\delta_l^*(m) = \begin{cases} \delta_l(m) : m \text{ and } q \text{ are relatively prime.} \\ 0 \quad\quad : m \text{ and } q \text{ are not relatively prime.} \end{cases} \text{, where } m \in \mathbb{Z}$$

Similarly, for each character $e \in \widehat{\mathbb{Z}^*(q)}$, we can extend e to all of $\mathbb{Z}$ via

$$\chi(m) = \begin{cases} e(m) : \text{if } m \text{ and } q \text{ are relatively prime.} \\ 0 \quad\quad : \text{ otherwise.} \end{cases} \text{, where } m \in \mathbb{Z}$$

Furthermore, we will define the extension of the trivial character of $\widehat{\mathbb{Z}^*(q)}$ to be χ_0, where $\chi_0(m) = 1$ if m and q are relatively prime, and 0 otherwise.

Now, we will restate our simplified δ_l Fourier series using the extended function δ_l^* and extended character χ.

$$\delta_l^*(m) = \frac{1}{\phi(q)} \sum_{\chi} \widehat{\chi(l)} \chi(m)$$

We will call this above equation **Claim 9 (Lemma 2.2, Pg 255)**.

Note from the how the extension of δ_l^* is defined, we see that for any prime number p such that $p \not\equiv l \bmod q$, $\delta_l^*(p) = 0$.

Hence,

$$\sum_{p \equiv l} \frac{1}{p^s} = \sum_p \frac{\delta_l^*(p)}{p^s}$$

By claim 9,

$$\sum_p \frac{\delta_l^*(p)}{p^s} = \frac{1}{\phi(q)} \sum_p \frac{1}{p^s} \sum_\chi \overline{\chi(l)} \chi(p) = \frac{1}{\phi(q)} \sum_\chi \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s}$$

We can break up the sum over all characters χ into the sum of a trivial character χ_0 component, and a non-trivial character component.

$$\frac{1}{\phi(q)} \sum_\chi \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s} = \frac{1}{\phi(q)} \overline{\chi_0(l)} \sum_p \frac{\chi_0(p)}{p^s} + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s}$$

First, $\overline{\chi_0(l)} = 1$.
Second, $\chi_0(p) = 1$ only if $(p, q) = 1$, and is 0 otherwise.

So with this further simplification, we have

$$\sum_{p \equiv l} \frac{1}{p^s} = \frac{1}{\phi(q)} \sum_{(p,q)=1} \frac{1}{p^s} + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s}$$

There are infinitely many primes p such that $(p, q) = 1$, therefore by claim 2,

$$\frac{1}{\phi(q)} \sum_{(p,q)=1} \frac{1}{p^s} \text{ is a divergent series when } s \to 1^+$$

This is why it matters that $\sum_p \frac{\chi(p)}{p^s}$ is bounded as $s \to 1^+$. The sum of a divergent series with a convergent series is always divergent. Since $\sum_p \frac{\chi(p)}{p^s}$ is a convergent series as $s \to 1^+$, and because there are only a finite amount of characters χ,

$$\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s} \text{ is also a convergent series as } s \to 1$$

So we can conlude that

$$\sum_{p \equiv l} \frac{1}{p}$$

is a divergent series, and therefore there must be infinite primes of the form $p = l + kq$, which proves Dirichlet's Theorem.

The only thing missing from this proof now is the supposition statement $\star$ which supposed that $L(1, \chi) \neq 0$.

The proof for $L(1, \chi) \neq 0$ will have to be split into 2 sections. Section 5 will tackle the case where χ is a complex Dirichlet character. Section 6 will be tackle the case where χ is a real Dirichlet character.

5. **L-function Properties 2. Complex Characters**

In this section, we are going to show that for a non-trivial complex valued Dirichlet character χ,

$$L(1,\chi) \neq 0.$$

This proof is takes a different approach than the proof for non-trivial real valued Dirichlet character, so that proof will be left for the next section. The main proof technique used for in this section is a proof by contradiction, and the two claims used in this proof will be proven at the end of the section.

Proof of Claim :

Let χ be a non-trivial complex valued Dirichlet character and χ_0 denote the trivial character.

Suppose towards a contradiction that $L(1,\chi) = 0$.

Then by claim 11 (i), $L(1,\overline{\chi}) = 0$ as well. Since χ is complex valued, $\chi \neq \overline{\chi}$. Lets call these two characters χ_1, χ_2.

$$\prod_{\chi} L(s,\chi) = L(s,\chi_0)L(s,\chi_1)L(s,\chi_2) \prod_{\chi \neq \chi_0,\chi_1,\chi_2} L(s,\chi)$$

By claim 10 (ii) and (iii),

$$L(s,\chi_0)L(s,\chi_1)L(s,\chi_2) \prod_{\chi \neq \chi_0,\chi_1,\chi_2} L(s,\chi) \leq C_1C_2|s-1|^2 \frac{C_3}{|s-1|} \prod_{\chi \neq \chi_0,\chi_1,\chi_2} L(s,\chi) = C_1C_2C_3|s-1| \prod_{\chi \neq \chi_0,\chi_1,\chi_2} L(s,\chi)$$

Therefore, as $s \to 1^+$, $\prod\limits_{\chi} L(s,\chi) \to 0$.

This last statement contradicts claim 10, which states $\prod\limits_{\chi} L(s,\chi) \leq 1$ when $s > 1$.

Hence, our original supposition was wrong, and $L(1,\chi) \neq 0$.

■

■Claims Used In Section

Claim 10 (Lemma 3.8 Pg. 266) : If $s > 1$, then

$$\prod_{\chi} L(s,\chi) \geq 1,$$

where the product is taken over all Dirichlet characters χ, the product is real valued.

Proof of Claim :

By claim 8, we have

$$L(s,\chi) = e^{\sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})}.$$

Then our product of L-functions is equal to

$$\prod_{\chi} L(s,\chi) = \prod_{\chi} e^{\sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})} = e^{\sum_\chi \sum_p log_1(\frac{1}{1-\chi(p)p^{-s}})}$$

Lets focus only on the exponent value.
By how log_1 is defined,

$$\sum_{\chi}\sum_{p} log_1(\frac{1}{1-\chi(p)p^{-s}}) = \sum_{\chi}\sum_{p}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\chi(p^k)}{p^{ks}} = \sum_{p}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\chi}\frac{\chi(p^k)}{p^{ks}}.$$

Recall that claim 9 stated

$$\delta_l^*(m) = \frac{1}{\phi(q)}\sum_{\chi}\widehat{\chi(l)}\chi(m)$$

Trivially, $\chi(1) = 1$.
Therefore, when $l = 1$,

$$\delta_1^*(m) = \frac{1}{\phi(q)}\sum_{\chi}\chi(m) \to \sum_{\chi}\chi(p^k) = \delta_1^*(p^k)\phi(q)$$

Therefore, our exponent for e simplifies to

$$\sum_{p}\sum_{k=1}^{\infty}\frac{1}{k}\delta_1^*(p^k)\phi(q)\frac{1}{p^{ks}}$$

Note, since $\frac{1}{k}, \delta_1(p^k), \phi(q)$, and $\frac{1}{p^{ks}}$ are all non-negative real values, the sum will have to be a real value ≥ 0.

Therefore, e to the power of this non-negative real value will be ≥ 1, and a real value.

■

Claim 11 (Lemma 3.9 Pg. 266) : The following three properties hold:

(i). If $L(1,\chi) = 0$, then $L(1,\overline{\chi}) = 0$.

(ii). If χ is non-trivial and $L(1,\chi) = 0$, then

$$|L(s,\chi) \leq C|s-1| \text{ when } 1 \leq s \leq 2$$

(iii). For the trivial Dirichlet character χ_0, we have

$$|L(s,\chi_0) \leq \frac{C}{|s-1|} \text{ when } 1 < s \leq 2$$

Proof of Claim :

(i). After expanding out $L(1,\overline{\chi})$, we find that $L(1,\overline{\chi}) = \overline{L(1,\chi)}$.

Hence, if $L(1,\chi) = 0$, then $L(1,\overline{\chi}) = \overline{L(1,\chi)} = \overline{0} = 0$.

■

(ii). Let $s \in [1,2]$.

By claim 7, $L(s,\chi)$ is continuously differentiable for $s > 0$.

Using the mean value theorem, there exists a $c \in [1,s]$ such that

$$L'(c,\chi) = \frac{L(s,\chi) - L(1,\chi)}{s-1} = \frac{L(s,\chi)}{s-1}$$

Since $L'(c,\chi)$ is a finite value, we can find a constant C such that $|L'(c,\chi)| \leq C$.

Therefore, $|L(s,\chi)| \leq C|s-1|$.

■

(iii). Let $s \in (1, 2]$.

By claim 5, $L(s, \chi_0) = (1 - p_1^{-s})(1 - p_2^{-s})...(1 - p_N^{-s})\zeta(s)$.

Since $(1 - p_1^{-s})(1 - p_2^{-s})...(1 - p_N^{-s})$ is only a finite product, we can use a finite number d to represent this value.

Recall $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which can be over approximated by

$$1 + \int_1^{\infty} \frac{1}{x^s} dx = 1 + \frac{1}{s-1}$$

Hence, $\zeta(s) \leq 1 + \frac{1}{s-1}$

This means that

$$|L(s, \chi_0)| \leq d|1 + \frac{1}{s-1}| = |\frac{ds}{s-1}|$$

Let $C = ds$, and we are done.

■

6. **L-function Properties 2. Real Characters**

In this section, we are going to show that for a non-trivial real valued Dirichlet character χ,

$$L(1,\chi) \neq 0.$$

Proof of Claim :

Let χ is a real Dirichlet non-trivial character.

Suppose towards a contradiction that $L(1,\chi) = 0$.

By claim 17 (i), $S_N \to \infty$ as $N \to \infty$.

By our supposition that $L(1,\chi) = 0$, and claim 17 (ii)., $S_N = O(1)$, a contradicton.

Therefore, our supposition was wrong and $L(1,\chi) \neq 0$.

■

■Claims Used In Section

Claims 12 to 16 below are the building blocks that leads to claim 17, the main claim used in the proof above.

Claim 12 (Proposition 3.10 Pg. 268) : If N is a positive integer, then:

$$(i). \sum_{n=1}^{N} \frac{1}{n} = \int_{1}^{N} \frac{dx}{x} + O(1) = ln(N) + O(1)$$

(ii). More precisely, there exists a real number γ, called Euler's constant, such that

$$\sum_{n=1}^{N} \frac{1}{n} = ln(N) + \gamma + O(\frac{1}{N})$$

Proof of Claim :

(i). This is the trivial case when you graph the series and integral, as $\int_{1}^{N} \frac{dx}{x}$ is an over appromixation of $\sum_{n=1}^{N} \frac{1}{n}$.

■

(ii). Let

$$\gamma_n = \frac{1}{n} - \int_{n}^{n+1} \frac{dx}{x}$$

Since $\frac{1}{x}$ is a decreasing function, $\frac{1}{n} \geq \int_{n}^{n+1} \frac{dx}{x} \geq \frac{1}{n+1}$, therefore

$$0 \leq \frac{1}{n} - \int_{n}^{n+1} \leq \frac{1}{n} - \frac{1}{n+1} \leq \frac{1}{n^2}$$

So $\gamma_n \leq \frac{1}{n^2}$, and because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convering harmonic series, $\sum_{n=1}^{\infty} \gamma_n$ will also converge.

Let $\gamma = \sum_{n=1}^{\infty} \gamma_n$, then

$$\sum_{n=1}^{N} \frac{1}{n} - \int_{1}^{N} \frac{dx}{x}$$

$$= \sum_{n=1}^{N} \frac{1}{n} - \int_{1}^{N+1} \frac{dx}{x} + \int_{N}^{N+1} \frac{dx}{x}$$

$$= \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \int_{n}^{n+1} \frac{dx}{x} + \int_{N}^{N+1} \frac{dx}{x}$$

$$= \sum_{n=1}^{N} \gamma_n + \int_{N}^{N+1} \frac{dx}{x}$$

$$= \gamma - \sum_{n=N+1}^{\infty} \gamma_n + \int_{N}^{N+1} \frac{dx}{x}$$

Since $\gamma_n \leq \frac{1}{n^2}$,

$$\sum_{n=N+1}^{\infty} \gamma_n \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_{N}^{\infty} \frac{1}{x^2} = O(\frac{1}{N})$$

Therefore,

$$\sum_{n=1}^{N} \frac{1}{n} - ln(N) = \sum_{n=1}^{N} \frac{1}{n} - \int_{1}^{N} \frac{dx}{x} = \gamma + O(1) + \int_{N}^{N+1} \frac{dx}{x}$$

Note that $\int_{N}^{N+1} \frac{dx}{x} = ln(1 + \frac{1}{N})$, so as $N \to \infty$, this becomes $O(\frac{1}{N})$.

■

Claim 13 (Proposition 3.11 Pg. 268) : If N is a positive integer, then:

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} = \int_{1}^{N} \frac{dx}{\sqrt{x}} + c' + O(\frac{1}{\sqrt{N}})$$

$$= 2\sqrt{N} + c + O(\frac{1}{\sqrt{N}})$$

Proof of Claim :

Let

$$c_n = \frac{1}{\sqrt{n}} - \int_{n}^{n+1} \frac{dx}{\sqrt{x}}$$

Since $\frac{1}{\sqrt{x}}$ is a decreasing function, $\frac{1}{\sqrt{n}} \geq \int_{n}^{n+1} \frac{dx}{\sqrt{x}} \geq \frac{1}{\sqrt{n+1}}$.
Therefore

$$0 \leq \frac{1}{\sqrt{n}} - \int_{n}^{n+1} \frac{dx}{\sqrt{x}} \leq \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

Let $f(x) = \frac{1}{\sqrt{x}}$. By the mean value theorem, there exists a $c \in [n, n+1]$ such that

$$-\frac{1}{2\sqrt{c^3}} = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$$

Since $c \geq n$, we can conclude that

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \leq \frac{1}{2\sqrt{n^3}}$$

This means that $\sum_{n=1}^{\infty} c_n$ is a converging series, and we will denote this infinite sum with c'.

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} - \int_{1}^{N} \frac{dx}{\sqrt{x}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sqrt{n}} - \int_{1}^{N+1} \frac{dx}{\sqrt{x}} + \int_{N}^{N+1} \frac{dx}{\sqrt{x}}$$

$$= \sum_{n=1}^{N} \frac{1}{\sqrt{n}} - \sum_{n=1}^{N} \int_{n}^{n+1} \frac{dx}{\sqrt{x}} + \int_{N}^{N+1} \frac{dx}{\sqrt{x}}$$

$$= \sum_{n=1}^{N} c_n + \int_{N}^{N+1} \frac{dx}{\sqrt{x}}$$

$$= c' - \sum_{n=N+1}^{\infty} c_n + \int_{N}^{N+1} \frac{dx}{\sqrt{x}}$$

The integral $\int_{N}^{N+1} \frac{dx}{\sqrt{x}}$ is less than or equal to $\frac{1}{\sqrt{N}}$, so we can denote this as $O(\frac{1}{\sqrt{N}})$.

Similarly,

$$\sum_{n=N+1}^{\infty} c_n \leq \sum_{n=N+1}^{\infty} \frac{1}{2\sqrt{n^3}} \leq \int_{n=N}^{\infty} \frac{dx}{2\sqrt{x^3}} = O(\frac{1}{\sqrt{N}})$$

Hence,

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} - \int_{1}^{N} \frac{dx}{\sqrt{x}} = c' + O(\frac{1}{\sqrt{N}})$$

Evaluating the integral in the equation above, we get $2\sqrt{N} - 2$.

By letting $c = c' + 2$, we obtain the second equality.

■

Hyperbolic Sums

If F is a function defined on two positive integers, then there are three ways to calculate

$$S_N = \sum \sum F(m, n),$$

where the sum if taken over all pairs of positive integers (m, n) such that $mn \leq N$.

Note that all three methods of summation gives the same sum.

(i). Along Hyperbolas

$$S_N = \sum_{1\leq k\leq N} \Big(\sum_{mn=k} F(m,n)\Big)$$

(ii). Vertically

$$S_N = \sum_{1\leq m\leq N} \Big(\sum_{1\leq n\leq \frac{N}{m}} F(m,n)\Big)$$

(i). Horizontally

$$S_N = \sum_{1\leq n\leq N} \Big(\sum_{1\leq m\leq \frac{N}{n}} F(m,n)\Big)$$

The Divisor Problem

For any positive integer k, let $d(k)$ denote the number of positive divisors of k.

For example, 5 only has 1 and 5 as divisors, so $d(5) = 2$.

10 has $1, 2, 5, 10$ as divisors, so $d(10) = 4$.

Note that

$$d(k) = \sum_{mn=k, 1\leq m,n} 1$$

Claim 14 (Proposition 3.12 Pg. 270) : If k is a positive integer, then:

$$\frac{1}{N}\sum_{k=1}^{N} d(k) = ln(N) + O(1)$$

More precisely,

$$\frac{1}{N}\sum_{k=1}^{N} d(k) = ln(N) + (2\gamma - 1) + O(\frac{1}{\sqrt{N}}), \text{ where } \gamma \text{ is Euler's Constant}$$

Proof of Claim :

Let

$$S_N = \sum_{k=1}^{N} d(k) = \sum_{k=1}^{N} \sum_{mn=k, 1\leq m,n} 1$$

This is sum above is simply summation along a hyperbola, where $F = 1$.

Next, we will try summing this vertically.

So

$$S_N = \sum_{k=1}^{N} d(k) = \sum_{1\leq m\leq N} \sum_{1\leq n\leq \frac{N}{m}} 1$$

$$= \sum_{1\leq m\leq N} [\frac{N}{m}] = \sum_{1\leq m\leq N} \frac{N}{m} + O(1), \text{ where } [\frac{N}{m}] \text{ is the floor function of } \frac{N}{m}$$

Therefore,

$$S_N = N \sum_{1\leq m\leq N} \frac{1}{m} + O(N)$$

By claim 12 (i), $\sum_{1\leq m\leq N} \frac{1}{m} = ln(N) + O(1)$.
After substituting this in, we get

$$S_N = N(ln(N) + O(1)) + O(N) = Nln(N) + O(N),$$

and the desired result

$$\frac{1}{N} S_N = ln(N) + O(1)$$

For the more refined estimate, we will break down the graph of $mn = N$ into three regions.

$I = \{1 \leq m \leq \sqrt{N}, \sqrt{N} \leq n \leq \frac{N}{m}\}$

$II = \{1 \leq m \leq \sqrt{N}, 1 \leq n \leq \sqrt{N}\}$

$III = \{\sqrt{N} \leq m \leq \frac{N}{n}, 1 \leq n \leq \sqrt{N}\}$

Let S_I, S_{II}, S_{III} denote the sums over these three regions. Note that the sums S_I and S_{III} are the exact same by the symmetry of these regions.

So the overall sum of the region, denoted by S_N is

$$S_N = S_I + S_{II} + S_{III} = 2S_I + S_{II} = 2(S_I + S_{II}) - S_{II}$$

We will first calculate the sum $S_I + S_{II}$ vertically.

$$S_I + S_{II} = \sum_{1\leq m\leq\sqrt{N}} (\sum_{1\leq n\leq \frac{N}{m}} 1) = \sum_{1\leq m\leq\sqrt{N}} \frac{N}{m} + O(1) = N(\sum_{1\leq m\leq\sqrt{N}} \frac{1}{m}) + O(\sqrt{N})$$

By claim 12 (ii), $\sum_{1\leq m\leq\sqrt{N}} \frac{1}{m} = ln(\sqrt{N}) + \gamma + O(\frac{1}{\sqrt{N}})$.
After substituting this in, we get

$$S_I + S_{II} = \frac{1}{2}Nln(N) + N\gamma + O(\sqrt{N})$$

On the other hand, S_{II} is simply

$$S_{II} = \sum_{1\leq m\leq\sqrt{N}} \sum_{1\leq n\leq\sqrt{N}} 1 = \sum_{1\leq m\leq\sqrt{N}} \sqrt{N} + O(1)$$

$$= (\sqrt{N} + O(1))\sqrt{N} + O(\sqrt{N}) = N + O(\sqrt{N})$$

Substituting these back into S_N, we get

$$S_N = 2(\frac{1}{2}Nln(N) + N\gamma + O(\sqrt{N})) - N + O(\sqrt{N}) = N[ln(N) + 2\gamma - 1 + O(\frac{1}{\sqrt{N}})]$$

and the desired result

$$\frac{1}{N} S_N = ln(N) + 2\gamma - 1 + O(\frac{1}{\sqrt{N}})$$

■

Claim 15 (Lemma 3.14 Pg. 273) Let χ be a non-trivial real Dirichlet character for some finite abelian group $\mathbb{Z}^*(q)$. Then

$$\sum_{n|k} \chi(n) \geq 0 \text{ for all } k$$

and

$$\sum_{n|k} \chi(n) \geq 1 \text{ if } k \text{ is a square.}$$

Proof of Claim :

Suppose first that k is simply the power of a prime number p.
Then the only divisors of k are $1, p, p^2, ..., p^a$, where $a \geq 0$.
Since χ is multiplicative, we get

$$\sum_{n|k} \chi(n) = \chi(1) + \chi(p) + \chi(p^2) + ...\chi(p^a) = 1 + \chi(p) + \chi(p)^2 + ... + \chi(p)^a$$

Since χ is a real valued non-trivial Dirichlet character, the only possible values of χ are ± 1 and 0.

If $\chi(p) = 1$, then $\sum_{n|k} \chi(n) = a + 1 \geq 1$.

If $\chi(p) = -1$, then $\sum_{n|k} \chi(n) = 0$ if a is odd, and 1 if a is even.

If $\chi(p) = 0$, then $\sum_{n|k} \chi(n) = 1$.

In any of those cases above, if k is p to an even power, the sum is ≥ 1.

Now, for an arbitrary $k = p_1^{a_1} p_2^{a_2} ... p_m^{a_m}$, where p_i are primes, the only divisors are of the form $p_1^{b_1} p_2^{b_2} ... p_m^{b_m}$, where $0 \leq b_i \leq a_i$ for all i.

Since χ is multiplicative, we get

$$\sum_{n|k} \chi(n) = \prod_{i=1}^{N} (\chi(1) + \chi(p_i) + \chi(p_i)^2 + ... + \chi(p_i)^{a_i}) = \prod_{i=1}^{N} (\sum_{n|{p_i}^{a_i}} \chi(m))$$

Since $\sum \chi(m) \geq 0$ whenever m is the power of a prime, $\sum_{n|k} \chi(n) \geq 0$.

If k is a square, then each a_i is even, hence each $\sum \chi(m) \geq 1$. Therefore $\sum_{n|k} \chi(n) \geq 1$.

■

Claim 16 (Lemma 3.15 Pg. 274) Let χ be a non-trivial real valued Dirichlet character for some abelian group $\mathbb{Z}^*(q)$. For all integers $0 < a < b$ we have

$$(i). \sum_{n=a}^{b} \frac{\chi(n)}{\sqrt{n}} = O(\frac{1}{\sqrt{a}})$$

$$(ii). \sum_{n=a}^{b} \frac{\chi(n)}{n} = O(\frac{1}{a})$$

Proof of Claim :

(i). Let

$$s_n = \sum_{1 \le k \le n} \chi(k), \text{ then } \chi(n) = s_n - s_{n-1}.$$

Utilizing this substituion, we get

$$\sum_{n=a}^{b} \frac{\chi(n)}{\sqrt{n}} = \frac{s_a - s_{a-1}}{\sqrt{a}} + \frac{s_{a+1} - s_a}{\sqrt{a+1}} + ... + \frac{s_b - s_{b-1}}{\sqrt{b}}$$

$$= \frac{s_a}{\sqrt{a}} - \frac{s_a}{\sqrt{a+1}} + \frac{s_{a+1}}{\sqrt{a+1}} - \frac{s_{a+1}}{\sqrt{a+2}} + ... + \frac{s_{b-1}}{\sqrt{b-1}} - \frac{s_{b-1}}{\sqrt{b}} + (\frac{s_b}{\sqrt{b}} - \frac{s_{a-1}}{\sqrt{a}})$$

$$= \frac{\sqrt{a}s_b - \sqrt{b}s_{a-1}}{\sqrt{b}\sqrt{a}} + \sum_{n=a}^{b-1} s_n[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}]$$

By claim 6, $|s_n| \le q$ for all n.

Therefore the first term above

$$\frac{\sqrt{a}s_b - \sqrt{b}s_{a-1}}{\sqrt{b}\sqrt{a}} = O(\frac{1}{\sqrt{a}})$$

To estimate the second term above, we will use the mean value theorem in the same way it was used in the proof of claim 7.

Define

$$g(x) = \frac{1}{\sqrt{x}}, \text{ then } g'(x) = -\frac{1}{2\sqrt{x^3}}$$

By the mean value theorem, there exists a $c \in [n, n+1]$ such that

$$g'(c) = -\frac{1}{2\sqrt{c^3}} = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}$$

and after multiplying by a negative 1,

$$\frac{1}{2\sqrt{c^3}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

Since $c \ge n$, this gives us the inequality

$$\frac{1}{2\sqrt{n^3}} \ge \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

Therefore,

$$|s_n(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})| \le \frac{q}{2\sqrt{n^3}}.$$

This means we can approximate

$$s_n[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}] = O(\frac{1}{\sqrt{n^3}}),$$

and our original sum becomes

$$\sum_{n=a}^{b} \frac{\chi(n)}{\sqrt{n}} = O(\sum_{n=a}^{\infty} \frac{1}{\sqrt{n^3}}) + O(\frac{1}{\sqrt{a}}).$$

Since

$$\sum_{n=a}^{\infty} \frac{1}{\sqrt{n^3}} = \int_a^{\infty} \frac{1}{\sqrt{x^3}} + O(1) = \frac{1}{2\sqrt{a}} + O(1) = O(\frac{1}{\sqrt{a}}),$$

we arrive at the desired result that

$$\sum_{n=a}^{b} \frac{\chi(n)}{\sqrt{n}} = O(\frac{1}{\sqrt{a}})$$

■

(ii). In a similar proof as (i), again we let

$$s_n = \sum_{1 \le k \le n} \chi(k), \text{ then } \chi(n) = s_n - s_{n-1}.$$

Utilizing this substituion, we get

$$\sum_{n=a}^{b} \frac{\chi(n)}{n} = \frac{s_a - s_{a-1}}{a} + \frac{s_{a+1} - s_a}{a+1} + ... + \frac{s_b - s_{b-1}}{b}$$

$$= \frac{s_a}{a} - \frac{s_a}{a+1} + \frac{s_{a+1}}{a+1} - \frac{s_{a+1}}{a+2} + ... + \frac{s_{b-1}}{b-1} - \frac{s_{b-1}}{b} + (\frac{s_b}{b} - \frac{s_{a-1}}{a})$$

$$= \frac{as_b - bs_{a-1}}{ba} + \sum_{n=a}^{b-1} s_n[\frac{1}{n} - \frac{1}{n+1}]$$

By claim 6, $|s_n| \le q$ for all n.

Therefore the first term above

$$\frac{as_b - bs_{a-1}}{ba} = O(\frac{1}{a})$$

To estimate the second term above, we will use the mean value theorem in the same way it was used in the proof of claim 7.

Define

$$g(x) = \frac{1}{x}, \text{ then } g'(x) = -\frac{1}{x^2}$$

By the mean value theorem, there exists a $c \in [n, n+1]$ such that

$$g'(c) = -\frac{1}{c^2} = \frac{1}{n+1} - \frac{1}{n}$$

and after multiplying by a negative 1,

$$\frac{1}{c^2} = \frac{1}{n} - \frac{1}{n+1}$$

Since $c \ge n$, this gives us the inequality

$$\frac{1}{n^2} \ge \frac{1}{n} - \frac{1}{n+1}$$

Therefore,

$$|s_n(\frac{1}{n} - \frac{1}{n+1})| \le \frac{q}{n^2}.$$

This means we can approximate

$$s_n[\frac{1}{n} - \frac{1}{n+1}] = O(\frac{1}{n^2}),$$

and our original sum becomes

$$\sum_{n=a}^{b} \frac{\chi(n)}{n} = O(\sum_{n=a}^{\infty} \frac{1}{n^2}) + O(\frac{1}{a}).$$

Since

$$\sum_{n=a}^{\infty} \frac{1}{n^2} = \int_a^{\infty} \frac{1}{x^2} + O(1) = \frac{1}{a} + O(1) = O(\frac{1}{a}),$$

we arrive at the desired result that

$$\sum_{n=a}^{b} \frac{\chi(n)}{n} = O(\frac{1}{a})$$

■

Claim 17 (Proposition 3.13 Pg. 272) Let χ be a non-trivial real valued Dirichlet character for some abelian group $\mathbb{Z}^*(q)$.
Let

$$F(m,n) = \frac{\chi(n)}{\sqrt{mn}},$$

and define

$$S_N = \sum\sum F(m,n),$$

where the sum is over all integers $m, n \geq 1$ that satisfy $mn \leq N$.

Then the following statements are true:

$$(i).\ S_N \geq c\ ln(N) \text{ for some constant } c$$
$$(ii).\ S_N = 2\sqrt{N}L(1,\chi) + O(1)$$

Proof of Claim :

(i). We will sum up S_N along hyperbolas.

$$S_N = \sum_{1\leq k\leq N} \sum_{mn=k} \frac{\chi(n)}{\sqrt{mn}} = \sum_{1\leq k\leq N} \frac{1}{\sqrt{k}} \sum_{n|k} \chi(n).$$

For this last sum, we can bound it below by restricting the k values to squares only. So

$$\sum_{1\leq k\leq N} \frac{1}{\sqrt{k}} \sum_{n|k} \chi(n) \geq \sum_{1\leq k\leq N, k=l^2} \frac{1}{\sqrt{k}} \sum_{n|k} \chi(n)$$

By claim 15, $\sum_{n|k} \chi(n) \geq 1$ when k is a square. So

$$\sum_{1\leq k\leq N, k=l^2} \frac{1}{\sqrt{k}} \sum_{n|k} \chi(n) \geq \sum_{1\leq k\leq N, k=l^2} \frac{1}{\sqrt{k}} = \sum_{1\leq l\leq [\sqrt{N}]} \frac{1}{l},$$

where $[\sqrt{N}]$ denotes the floor function of $\sqrt{N}$.

By claim 12,

$$\sum_{1\leq l\leq [\sqrt{N}]} \frac{1}{l} = \frac{1}{2} ln(N) + O(1) \geq c\ ln(N) \text{ for some constant } c.$$

Hence,

$$S_N \geq c\ ln(N) \text{ for some constant } c$$

■

(ii). Similar to the proof of cliam 14, we will write $S_N = S_I + S_{II} + S_{III}$. Where

$I = \{1 \le m \le \sqrt{N}, \sqrt{N} \le n \le \frac{N}{m}\}$,

$II = \{1 \le m \le \sqrt{N}, 1 \le n \le \sqrt{N}\}$,

$III = \{\sqrt{N} \le m \le \frac{N}{n}, 1 \le n \le \sqrt{N}\}$,

and S_I, S_{II}, S_{III} denotes the sum over these three regions. Note that the sums S_I and S_{III} are the exact same by the symmetry of these regions.

We will evaluate S_I by summing vertically, and $S_{II} + S_{III}$ by summing horizontally.

Summing vertically,

$$S_I = \sum_{1 \le m \le \sqrt{N}} \sum_{\sqrt{N} \le n \le \frac{N}{m}} \frac{\chi(n)}{\sqrt{mn}} = \sum_{1 \le m \le \sqrt{N}} \frac{1}{\sqrt{m}} \sum_{\sqrt{N} \le n \le \frac{N}{m}} \frac{\chi(n)}{\sqrt{n}}$$

By claim 16 (i),

$$\sum_{1 \le m \le \sqrt{N}} \frac{1}{\sqrt{m}} \sum_{\sqrt{N} \le n \le \frac{N}{m}} \frac{\chi(n)}{\sqrt{n}} = \sum_{1 \le m \le \sqrt{N}} \frac{1}{\sqrt{m}} O(N^{-\frac{1}{4}}) = O(N^{-\frac{1}{4}}) \sum_{1 \le m \le \sqrt{N}} \frac{1}{\sqrt{m}}$$

By claim 13,

$$O(N^{-\frac{1}{4}}) \sum_{1 \le m \le \sqrt{N}} \frac{1}{\sqrt{m}} = O(N^{-\frac{1}{4}})[2N^{\frac{1}{4}} + c + O(N^{-\frac{1}{4}})]$$

$$= O(N^{-\frac{1}{4}} N^{\frac{1}{4}}) + O(N^{-\frac{1}{4}}) + O(N^{\frac{1}{8}}) = O(1)$$

Next, summing horizontally,

$$S_{II} + S_{III} = \sum_{1 \le n \le \sqrt{N}} \sum_{1 \le m \le \frac{N}{n}} \frac{\chi(n)}{\sqrt{mn}} = \sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{\sqrt{n}} \sum_{1 \le m \le \frac{N}{n}} \frac{1}{\sqrt{m}}$$

By claim 13 again,

$$\sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{\sqrt{n}} \sum_{1 \le m \le \frac{N}{n}} \frac{1}{\sqrt{m}} = \sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{\sqrt{n}} [2\sqrt{\frac{N}{n}} + c + O(\sqrt{\frac{n}{N}})]$$

$$= 2\sqrt{N} \sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{n} + c \sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{\sqrt{n}} + O(\frac{1}{\sqrt{N}} \sum_{1 \le n \le \sqrt{N}} 1)$$

Trivially,

$$O(\frac{1}{\sqrt{N}} \sum_{1 \le n \le \sqrt{N}} 1) = O(1)$$

By claim 16 (i).

$$c \sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{\sqrt{n}} = c\ [O(1)] = O(1)$$

For the first term in the sum above,

$$2\sqrt{N} \sum_{1 \le n \le \sqrt{N}} \frac{\chi(n)}{n} = 2\sqrt{N} \sum_{1}^{\infty} \frac{\chi(n)}{n} - 2\sqrt{N} \sum_{\sqrt{N}+1}^{\infty} \frac{\chi(n)}{n}$$

From how it was defined,

$$2\sqrt{N}\sum_{1}^{\infty}\frac{\chi(n)}{n} = 2\sqrt{N}L(1,\chi).$$

By claim 16 (i),

$$2\sqrt{N}\sum_{\sqrt{N}}^{\infty}\frac{\chi(n)}{n} = 2\sqrt{N}O(\frac{1}{\sqrt{N}+1} = O(1).$$

Therefore, combining all the pieces we have,

$$S_N = 2\sqrt{N}L(1,\chi) + O(1)$$

■