MATH 562 - CSULB Fall 19'
Notes on Complex Analysis

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Contents

1 Introduction to Complex Numbers

Definitions.

Define

$$\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(x, y)|x, y \in \mathbb{R}\}$$

to be the set of complex numbers or ordered pairs. This set comes with natural operations $+, \cdot$ defined via

$$(a, b) + (c, d) = (a + c, b + d) \quad (1)$$
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc) \quad (2)$$

For every $a, b, c, d \in \mathbb{R}$ turning $\mathbb{C}$ into a field denoted $(\mathbb{C}, +, \cdot)$ with additive identity and multiplicative identities

$$+_{id} = (0, 0)$$
$$\cdot_{id} = (1, 0).$$

<back2top>

2 $\mathbb{C}$ as a field

Since we know $(\mathbb{R}, +, \cdot)$ is a field, there is a natural field homomorphism

$$F : (\mathbb{R}, +, \cdot) \to (\mathbb{C}, +, \cdot)$$

defined via

$$F(x) = (x, 0)$$

Note F is 1-1 so it also has an inverse.

Also note a field homomorphism is an operation preserving map between two fields

Thus, it is natural to write $(a, 0)$ for some $a \in \mathbb{C}$. Furthermore, we know for some $(a, b) \in \mathbb{C}$ that we can write

$$\begin{aligned}(a, b) &= (a, 0) + (0, b) \\ &= a + b(0, 1)\end{aligned}$$

Where $(0, 1)$ is i and so it makes sense to write

$$\mathbb{C} = \{a + bi|a, b \in \mathbb{R}\}.$$

To see why $i^2 = -1 = (-1, 0)$ we compute $(-1, 0) \cdot (-1, 0)$ using the operation defined in 1(2). Furthermore note that

$$a + bi = c + di$$

if and only if
$$(a = c) \wedge (b = d).$$
And if for $z \in \mathbb{C}$, we can write $z = a + bi$ where $\text{Re}(z) = a$ and $\text{Im}(z) = b$ denote the *real* and *imaginary* parts of z, respectively.

<back2top>

3 $\mathbb{C}$ as a Metric Space

It turns out that $\mathbb{C}$ can be turned into a metric space. First we let two elements $z, w \in C$ be expressed as
$$z = a + bi, w = c + di$$
Then let $d(z, w)$ denote distance or the *metric function*:
$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$
Note that d enjoys the Triangle Inequality. That is, for every $z, w, v \in \mathbb{C}$,
$$d(z, w) \leq d(z, v) + d(v, w).$$
We can equip $\mathbb{C}$ with a norm on its elements. That is, for any $z \in \mathbb{C}$ we can define $|z|$ as distance from $(0, 0)$ to $z = a + bi$, given via
$$|z| = \sqrt{a^2 + b^2}$$
Then we have another version of the Triangle Inequality:
$$|w + z| \leq |w| + |z|.$$
As well the norm is multuplicative. That is,
$$|zw| = |z||w|.$$

<back2top>

4 The complex conjugate

Cool fact,
$$a^2 + b^2$$
is actually a difference of squares in $\mathbb{C}$. To see this let $z, w \in \mathbb{C}$, then
$$z^2 + w^2 = (z + w)(z - w)$$
This is clearly not true in $\mathbb{R}$. To make some sense of this we define the *complex conjugate* of $z \in \mathbb{C}$ as
$$\bar{z} = a - bi \quad ; \quad z = a + bi \in \mathbb{C}.$$

The complex conjugate enjoys some nice properties:

$$\text{(i)} \qquad z\overline{z} = a^2 + b^2 \tag{3}$$
$$= |z| \tag{4}$$

$$\text{(ii)} \qquad |z| = |\overline{z}| \tag{5}$$

$$\text{(iii)} \qquad z + \overline{z} = 2\text{Re}(z). \tag{6}$$

<*back2top*>

5 Inequality Theory

A binary relation, $\leq$ is a *total ordering* on a set X if the following holds

$$\text{(i)} \qquad a \leq b \vee b \leq a \qquad \text{connexity} \tag{7}$$

$$\text{(ii)} \qquad (a \leq b) \wedge (b \leq c) \Rightarrow a \leq c \qquad \text{transitivity} \tag{8}$$

$$\text{(iii)} \qquad (a \leq b \wedge c \leq 0) \Rightarrow ac \leq bc \qquad \text{anti-symmetry} \tag{9}$$

It turns out, $\mathbb{C}$ is not totally ordered. We verify this with proof.

Proof. Let assume towards a contradiction that there exists some total ordering $\succeq$, on $\mathbb{C}$. Then $\succeq$ satisfies $5(7), 5(8), 5(9)$. So we let

$$i \succeq 0$$

Multiplying both side by i together with $i^1 = -1$ we see that

$$-1 \succeq 0$$

Adding 1 to both sides yields

$$0 \succeq 1.$$

Multiplying both sides again by i we see that

$$0 \succeq i.$$

Thus we have

$$(i \succeq 0) \wedge (0 \succeq i) \Rightarrow 0 = i$$

a contradiction and therefore $\mathbb{C}$ has no total ordering. □

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6 Polar Coordinates

Let $z \in \mathbb{C}$ be given. One could find another alternate (canonical) form for z other that rectangular. If we let

$$z = a + bi \quad ; \quad a, b \in \mathbb{R},$$

We could then let z be represented by the ordered pair

$$(r, \theta) \quad ; \quad r = |z|$$

With $\theta \in \arg z$ where

$$\arg z = \{\theta + 2\pi n | n \in \mathbb{Z}\}.$$

A few properties of polar coordinates are

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$(re^{i\theta})^n = r^n e^{in\theta}$$

where $e^{i\theta} = \cos\theta + i \sin\theta$.

<back2top>

7 Sequences & Series

Let $\{a_1, a_2, ...\}$ be a sequence in $\mathbb{C}$. We say that the series $\sum_{i=1}^{n} a_i$ *converges* to some $A \in \mathbb{C}$ if for any given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $k \geq N$, then

$$\Big| \sum_{n=1}^{k} a_n - A \Big| < \epsilon.$$

If $\sum |a_n|$ converges, we say the series *converges absolutely*. Note that it is clear that if a series converges absolutely, then it converges in the usual sense as well. We say that a sequence is *Cauchy* if for any given $\epsilon > 0$, there exists and $N \in \mathbb{N}$ such that if $m, n \geq N$, then

$$|a_n - a_m| < \epsilon.$$

Corollary: A sequence (in $\mathbb{C}$, well also in $\mathbb{R}$) converges if and only if it is Cauchy.

We now define the *tail−end* of a series to be given via the following bi-conditional:

$$S_N = \sum_{n=1}^{N} a_n$$

converges if and only if

$$t_m = \sum_{n=m}^{\infty} a_n \to 0$$

as $n \to \infty$.

Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof. Let $\epsilon > 0$ be given. Assume that $\sum |a_n|$ converges. Then the tail-end goes to 0 as $n \to \infty$. That is,

$$t_m = \sum_{n=m}^{\infty} |a_n| \to 0.$$

Let $N \in \mathbb{N}$ be such that if $m \geq N$, then

$$\begin{aligned} \big|\sum_{n=m}^{\infty} a_n - 0\big| &= \big|\sum_{n=m}^{\infty} a_n\big| \\ &< \epsilon. \end{aligned}$$

Now consider the series

$$S_m = \sum_{n=1}^{m} a_n.$$

I claim S_m is a Cauchy sequence. Let $m, k \geq N$ and without any loss of generality say $m > k$. Then

$$\begin{aligned} |S_m - S_k| &= \big|\sum_{n=k+1}^{m} a_n\big| \\ &\leq \sum_{n=k+1}^{m} |a_n| \\ &\leq \sum_{n=k+1}^{\infty} |a_n| \\ &< \epsilon. \end{aligned}$$

by the tail-end and thus S_m is Cauchy as needed, hence convergent.
Next, let $\{a_n\}$ be a sequence in $\mathbb{R}$. We *define*

$$\begin{aligned} \limsup a_n &= \lim_{n\to\infty} \sup\{a_n, a_{n+1} ...\} \\ \liminf a_n &= \lim_{n\to\infty} \inf\{a_n, a_{n+1} ...\}. \end{aligned}$$

Example 7.1

Take $a_n = (-1)^n$. Then

$$\begin{aligned} \limsup a_n &= 1 \\ \liminf a_n &= -1. \end{aligned}$$

Example 7.2

Take $a_n = ((-1)^n + 1)n$. Then

$$a_n = \begin{cases} 0 & ; n \quad \text{odd} \\ 2n & ; n \quad \text{even} \end{cases}$$

And thus

$$\limsup a_n = \infty$$
$$\liminf a_n = 0.$$

□

A *power series* is a series of the form

$$\sum_{n\in\mathbb{N}} a_n(z-a)^n,$$

where the sequence of coefficients comes from $\mathbb{C}$.

Example 7.3

The *geometric* series

$$\sum_{n\in\mathbb{N}} z^n \to \frac{1}{1-z}.$$

For when $z| < 1$. If however $|z| > 1$, then the series is said to diverge.

Big Theorem (Conway III.1.3): For a given power series

$$\sum_{n\in\mathbb{N}} a_n(z-a)^n,$$

and $R \in [0, \infty]$ if we define

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}},$$

then

(a) If $|z-a| < R$, then the series converges absolutely.

(b) If $|z-a| > R$, then the terms become unbounded and the series diverges.

(c) If $r \in (0, R)$, then the series converges uniformly on $\{z : |z-a| \le r\}$. Namely, R is the only number satisfying (a) and (b).

Proof. For (a), we take $a = 0$. Then if $|z| < R$, we show

$$\sum_{n\in\mathbb{N}} a_n z^n$$

converges. Then if $|w-a| < R$,

$$\sum_{n\in\mathbb{N}} a_n(w-a)^n$$

converges as well. As $|z| < R$, then there exists $r > 0$ such that

$$|z| < r < R.$$

I claim for this $r \in (0, R)$ there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$,

$$|a_n|^{\frac{1}{n}} \leq \frac{1}{r}.$$

We know however that

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}},$$

and since $r < R$, then $\frac{1}{r} > \frac{1}{R}$ and so $\frac{1}{r} - \frac{1}{R} > 0$ and so we can take $\epsilon = \frac{1}{r} - \frac{1}{R}$. Then there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$, then

$$\left| \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, ...\} - \frac{1}{R} \right| < \frac{1}{r} - \frac{1}{R}.$$

Thus $\sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, ...\} < \frac{1}{r}$. By definition of sup we have that

$$|a_n|^{\frac{1}{n}} < \frac{1}{r}.$$

Then if $n \geq N$ we obtain

□

<back2top>