Free Calculus Supplement

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Contents

1 Derivatives

- **Power Rule:** If
$$f(x) = ax^n,$$
then
$$f'(x) = nax^{n-1}.$$
- **Product Rule** If f can be written as the product of two functions,
$$f(x) = g(x)h(x),$$
then
$$f'(x) = g'(x)h(x) + g(x)h'(x).$$
- **Quotient Rule** If f can be written as the quotient of two functions,
$$f(x) = \frac{g(x)}{h(x)},$$
where $h(x) \neq 0$, then
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}.$$
- **Chain Rule** If f can be written as a function inside another function, say
$$f(x) = g(h(x)),$$
then
$$f'(x) = g'(h(x))h'(x).$$
- **Exponential** e If f is given as e raised to some function, i.e.,
$$f(x) = e^{g(x)},$$
then
$$f'(x) = g'(x)e^{g(x)}$$
- **Natural Log** If f is the natrual log of a function, i.e.,
$$f(x) = \ln(g(x)),$$
then
$$f'(x) = \frac{g'(x)}{g(x)}.$$
- **Logarithm** If f is expressed as the logarithm, that is,
$$f(x) = \log_a g(x),$$
then
$$f'(x) = \frac{g'(x)}{g(x)\ln a}$$

- **General Exponential** If f is written as a general exponential, that is,

$$f(x) = a^{g(x)},$$

then

$$f'(x) = g'(x)a^{g(x)} \ln a.$$

- **Inverses** If $f(x)$ and $g(x)$ are inverses of one another, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

- **Constants** If f is any constant function it has derivative 0.
- **Absolute Value** If f is the absolute value of a function, i.e.,

$$f(x) = |g(x)|,$$

then

$$f'(x) = g'(x)\frac{g(x)}{|g(x)|}.$$

- **Square root** If f is $\sqrt{x}$, then

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Trigonometric functions - Derivatives:

- **Sin and cosine** If f is $\sin(g(x))$, then

$$f'(x) = \cos(g(x))g'(x).$$

And if f is $\cos(g(x))$, then

$$f'(x) = -\sin(g(x))g'(x).$$

Using the fact that $\tan x = \frac{\sin x}{\cos x}$, we find that the derivative of $\tan g(x)$, then

$$f'(x) = \sec^2(g(x))g'(x).$$

The rest can be obtained from these three.

- **Inverse sin** If f is arcsin$x$, then

$$f'(x) = \frac{1}{\sqrt{1-x^2}}.$$

- **Inverse cos** If f is arccos$x$, then

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

- **Inverse tan** If f is arctan$x$, then

$$f'(x) = \frac{1}{1+x^2}.$$

- **Inverse cot** If f is arccot$x$, then

$$f'(x) = -\frac{1}{1+x^2}.$$

- **Secant** If $f(x) = \sec x$, then

$$f'(x) = \sec x \tan x.$$

- **Cosecant** If $f(x) = \csc x$, then

$$f'(x) = -\csc x \cot x.$$

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2 Integrals

- If $f(x) = \int ax^n dx$, then its anti-derivative is given by

$$F(x) = \frac{a}{n+1}x^{n+1} + C,$$

  where C is some constant, it may or may not be zero.

- If an integral is given in terms two parts as in integrating by parts, we have

$$uv - \int vdu$$

  Where u is the first in LIATE.

  To see an example, we determine the integral of $(\ln x)^2$, click here!

- If $f(x) = \frac{1}{x}$, then

$$F(x) = \int \frac{1}{x}dx = \ln|x|.$$

- If $f(x) = a^x$, then

$$F(x) = \int a^x dx = \frac{a^x}{\ln a}.$$

- If $f(x) = \ln x$, then via integration by parts we have that

$$\begin{aligned} F(x) &= \int f(x)dx \\ &= x\ln x - x + C \\ &= x(\ln x - 1) + C. \end{aligned}$$

Trig Integrals:

- If $f(x) = \cos x dx$ then

$$F(x) = \int \cos x dx = \sin x.$$

- If $f(x) = \sin x dx$ then

$$F(x) = \int \sin x dx = -\cos x.$$

- If $f(x) = \tan x$, then

$$F(x) = \int \tan x dx = \ln|\sec x|.$$

- If $f(x) = \cot x$, then

$$F(x) = \int \cot x dx = \ln|\sin x|.$$

- If $f(x) = \sec x$, then

$$F(x) = \int \sec x dx = \ln|\sec x + \tan x|.$$

- If $f(x) = \csc x$, then

$$F(x) = \int \csc x dx = \ln|\csc x - \cot x|.$$

- Inverse sin

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin(\frac{x}{a}).$$

- Inverse cos

$$-\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin(\frac{x}{a}).$$

- Inverse tangent

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan(\frac{x}{a})$$

And for an application of Integration by Parts, if $f(x) = (\ln x)^2$ and we wish to find

$$\begin{aligned} F(x) &= \int f(x) dx \\ &= \int (\ln x)^2 dx. \end{aligned}$$

Set

$$x = e^y, \qquad (*)$$

then

$$dx = e^y dy \qquad (**).$$

Then we have

$$\begin{aligned} F(x) &= \int f(x) dx \\ &= \int (\ln x)^2 dx \\ &= \int (\ln(e^y))^2 e^y dy \\ &= \int \ln e^{y^2} e^y dy \\ &= \int y^2 e^y dy. \end{aligned}$$

And here we apply integration by parts and set

$$u = y^2, dv = e^y$$

and thus

$$du = 2ydy, v = e^y$$

and we get

$$\int y^2 e^y dy = y^2 e^y - 2^y dy.$$

Applying IBP once more we get

$$y^2 e^y - 2ye^y + 2e^y + C$$

and substituting back using $(*)$ and $(**)$ we have

$$F(x) = x(\ln x)^2 - 2x \ln x + 2x + C.$$

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3 Theorems

- **Mean Value Theorem** If f is continous on $[a, b]$ then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- **Intermediate Value Theorem** If f is continuous on $[a, b]$ such that for any z strictly between $f(a)$ and $f(b)$ there exists a $c \in (a, b)$ such that $f(c) = z$.

- **Fundamental Theorem of Calculus** Let $f : [a, b] \to \mathbb{R}$ be continuous. For every $x \in [a, b]$ let $F(x)$ be defined as

$$F(x) = \int_a^x f(t).$$

  Then f is uniformly continuous on $[a, b]$, differentiable on (a, b), and

$$F'(x) = f(x)$$

  for every $x \in [a, b]$.

- **FTC Cor.** If $f : [a, b] \to \mathbb{R}$ is continuous and F is the anti-derivative, then

$$\int_a^b f(x)dx = F(b) = F(a).$$

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