Real Analysis Select Solutions

MyMathYourMath.com
Solutions by: Hossien Sahebjame

2021

Contents

1. Monotone Convergence Theorem (MCT) for Measurable sets.
2. Some outer measure.
3. Measure of a continuous graph
4. Fatou's Lemma $\Rightarrow$ MCT
5. Application of MCT for sets
6. Construction of closed set
7. Proving Bounded Convergence via Egorov
8. Approximate measurable via Closed and Compact (sub)sets.
9. Crazy Integral
10. Countable Sub-Additivity for outer Measure
11. Lipschitz preserves measure zero
12. Construction of closed subset of $[0,1] \setminus \mathbb{Q}$
13. Tchebyshev inequality
14. Convergence: Measure VS. Point-wise
15. Continuous functions are Borel Measurable
16. Showing $C([-1,1])$ is not complete.
17. Converge in measure and $L^1(\mathbb{R})$ but not a.e. (& MORE)
18. Cauchy sequence of simple functions
19. Existence of simple functions converge to f.
20. Examples of integrability
21. Approximate finite measure sets via open set
22. Fatou Implies DCT
23. If $f \in L^1$ so is $f_h - f$

1. MCT for Sets

We will prove the following two theorems, the latter depends on the first.
Let $E_1, E_2, ...$ be a sequence of measurable sets. If they are an increasing sequence, i.e.,

$$E_n \subset E_{n+1},$$

then we have that

$$\lim_{n\to\infty} m(E_n) = m(\bigcup_{n=1}^{\infty} E_n).$$

Part II states given $E_1, E_2, ...$, a sequence of measurable functions, if $m(E_1) < \infty$ and they are decreasing, i.e.,

$$E_n \supset E_{n+1},$$

then we have that

$$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=1}^{\infty} E_n).$$

Proof. For the first one, let $G_1 = E_1$ and for every $n \geq 2$ let $G_n = E_n \setminus E_{n+1}$. Then we have that $\bigcup E_n = \bigcup G_n$ and we can compute

$$\begin{aligned}
m(\bigcup_{n=1}^{\infty} E_n) &= m(\bigcup_{n=1}^{\infty} G_n) && \text{monotonicity} \\
&= \sum_{n=1}^{\infty} m(G_n) && \text{as the } G_n \text{ are disjoint, additivity} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} m(G_n) && \text{sequence of partials sums} \\
&= \lim_{N\to\infty} m(\bigcup_{n=1}^{N} G_n) && \text{countable (finite) additivity, as } G_n \text{ disjoint} \\
&= \lim_{N\to\infty} m(\bigcup_{n=1}^{N} E_n) && \text{since } \bigcup E_n = \bigcup G_n \\
&= \lim_{N\to\infty} m(E_N) && \text{as the } E_n \text{ are increasing, } E_N \text{ is maximal}
\end{aligned}$$

as needed.
For the latter, first note that as the E_n are decreasing, then

$$E_1 \setminus E_n \subset E_1 \setminus E_{n+1}.$$

are increasing thus we can apply our part I. Furthermore, note that we have by set theory that

$$\begin{aligned}
\bigcup E_1 \setminus E_n &= \bigcup E_1 \cap E_n^c \\
&= E_1 \cap \bigcup E_n^c \\
&= E_1 \cap (\bigcap E_n)^c \\
&= E_1 \setminus \bigcap E_n.
\end{aligned}$$

Thus we can compute

$$\begin{aligned}
m(E_1) - m(\bigcap E_n) &= m(E_1 \setminus \bigcap E_n) && \text{finite measure is additive} \\
&= m(\bigcup E_1 \setminus E_n) && \text{by set theory} \\
&= \lim_{n\to\infty} m(E_1 \setminus E_n) && \text{by MCT for increasing sets} \\
&= m(E_1) - \lim_{n\to\infty} m(E_n) && m(E_1) \text{ not dependant on } n
\end{aligned}$$

As $m(E_1) < \infty$ we can subtract it from both sides to get the desired result. □

<back2top>

2. Some key outer measures

Prove $m_*([0,1]) = 1$ where m_* is outer measure.

Proof. First we show $m_*([0,1]) = 1$ by showing $m_*([0,1]) \leq 1$ and $m_*([0,1]) \geq 1$. First note that if $\bigcup_j I_j$ is a covering of $[0,1]$ by closed sets, as $[0,1]$ is closed itself we have that

$$I_1 := [0,1], I_j = \emptyset; \forall j \geq 2.$$

Then we can compute

$$\begin{aligned} m_*([0,1]) &= \inf\{\sum_j |I_j|\} && \text{definition of outer measure} \\ &= \inf\{\sum |[0,1]|\} && \text{as } I_j = \emptyset \text{ ; } \forall j \geq 2 \\ &\leq |[0,1]| && \text{definition of infimum} \\ &= 1 && \text{volume of the unit interval.} \end{aligned}$$

And on the other hand let $\{I_j\}_j$ be a covering of $[0,1]$ by closed sets. That is,

$$[0,1] \subset \bigcup_j I_j.$$

And thus we can compute

$$\begin{aligned} \sum_j |I_j| &\geq |\bigcup_j I_j| && \text{countable sub-additivity} \\ &\geq |[0,1]| && \text{monotonicity} \\ &= 1 && \text{volume of the unit interval} \end{aligned}$$

Thus $\inf\{\sum_j |I_j|\} \geq 1$ and we can conclude that

$$m_*([0,1]) = 1$$

as needed. □

Prove $m_*([0,1] \cap \mathbb{Q}) = 0$.

Proof. Note that

$$[0,1] \cap \mathbb{Q} \subset \mathbb{Q}.$$

Thus by monotonicity of the outer measure we have that

$$m_*([0,1] \cap \mathbb{Q}) \leq m_*(\mathbb{Q}).$$

I claim $m_*(\mathbb{Q}) = 0$. We show this first showing singletons in $\mathbb{R}$ have measure zero, then that countable subsets of $\mathbb{R}$ have measure zero. Let $A = \{a\}$. We show $m_*(A) = 0$. For any $\epsilon > 0$ note that

$$A \subset (a - \epsilon, a + \epsilon).$$

Then we can apply monotonicity and thus compute

$$\begin{aligned} m_*(A) &\le m_*((a-\epsilon, a+\epsilon)) \\ &= |(a-\epsilon, a+\epsilon)| \\ &= 2\epsilon \end{aligned}$$

and we are done as ϵ was arbitrary. Now we show that any countable subset of $\mathbb{R}$ has measure 0. If $B \subset \mathbb{R}$ is countable, then we can write B as a union of countable many singletons. That is,

$$B = \bigcup_{n\in\mathbb{N}} \{b_n\}.$$

Then we can compute

$$\begin{aligned} m(B) &= m(\bigcup_{n\in\mathbb{N}} \{b_n\}) \\ &\le \sum_{n\in\mathbb{N}} m(\{b_n\}) \\ &= 0 + 0 + 0 + ... \\ &= 0. \end{aligned}$$

Since $\mathbb{Q} \subset \mathbb{R}$ is countable, we conclude that $m_*(\mathbb{Q}) = 0$ and therefore $m_*([0,1] \cap \mathbb{Q}) = 0$ by monotonicity. □

Prove $m_*([0,1] \setminus \mathbb{Q}) = 1$.

Proof. Again, note that as we have the set containment

$$([0,1] \setminus \mathbb{Q}) \subset [0,1],$$

we have by monotonicity that

$$\begin{aligned} m_*([0,1] \setminus \mathbb{Q}) &\le m_*([0,1]) \\ &= 1. \end{aligned}$$

So we are left to show $m_*([0,1] \setminus \mathbb{Q}) \ge 1$. Observe that we can write the unit interval as

$$[0,1] = ([0,1] \setminus \mathbb{Q}) \cup ([0,1] \cap \mathbb{Q})$$

Then we can apply countable sub-additivity and compute

$$\begin{aligned} 1 &= m_*([0,1]) && \text{by part 1} \\ &= m_*(([0,1] \setminus \mathbb{Q}) \cup ([0,1] \cap \mathbb{Q})) && \text{by observation above} \\ &\le m_*([0,1] \setminus \mathbb{Q}) + m_*(([0,1] \cap \mathbb{Q}) && \text{countable sub-additivity} \\ &= m_*([0,1] \setminus \mathbb{Q}) + 0 && \text{by part 2} \end{aligned}$$

Hence we have $m_*([0,1] \setminus \mathbb{Q}) = 1$. □

<back2top>

3. Measure of continuous graph

Prove continuous functions have a measure zero graph.

Proof. First we show that if

$$f : [a, b] \to \mathbb{R}$$

is continuous, then the graph

$$\Gamma(f) = \{(x, f(x)) : x \in [a, b]\},$$

has measure zero. First note that as f is continuous over a compact interval then f need be uniformly continuous on the entire interval. So then we can let

$$P = \{x_0, x_1, ..., x_n\}$$

be a partition of our interval such that for any given $\epsilon > 0$ we can always find a $\delta > 0$ such that if

$$|x_i - x_{i-1}| < \delta,$$

then

$$|f(x_i) - f(x_{i-1})| < \epsilon.$$

As $[a, b]$ is compact, f attains its optimal points. That is, we can define

$$m_i := \min_{x \in [x_i, x_{i-1}]} f(x); M_i := \max_{x \in [x_i, x_{i-1}]} f(x).$$

And so we have the following set containment

$$G(f) \subseteq \bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [m_i, M_i].$$

Then by monotonicity we get

$$\begin{aligned} m(G(f)) &\leq m(\bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [m_i, M_i]) \\ &= \sum_{i=1}^{n} |x_i - x_{i-1}| \cdot |M_i - m_i| \\ &< (b - a)\epsilon \end{aligned}$$

Where $\delta = \frac{b-a}{n}$ makes this work.
Next we show if

$$f : \mathbb{R} \to \mathbb{R}$$

is a continuous function, then it has a graph of measure zero. Let

$$\Gamma(f) = \{(x, f(x) : x \in \mathbb{R}\}.$$

As the map $x \mapsto -x$ preserves area of rectangles, it suffices to suppose we are non-negative. Thus we can write

$$\Gamma(f) = \{(x, f(x)) : x \in [n, n + 1]\}$$

Then by the earlier part taking $a = n, b = n + 1$ we have that the graphs measure is a countable union of measure zero which has measure zero. □

<back2top>

4. Fatou implies MCT

Prove MCT via Fatou's Lemma. We first state the two results:

> Fatou's Lemma: If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative measurable functions converging to some measurable function f, then
>
> $$\int f \leq \lim_{n\to\infty} \inf \int f_n.$$
>
> Monotone Convergence Theorem: If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative measurable functions increasing to some measurable function f, then
>
> $$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. So let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions converging to f. By Fatou's Lemme we have that

$$\int f \leq \lim_{n\to\infty} \inf \int f_n.$$

Since $f_n \nearrow f$ we have

$$\int f_n \leq \int f$$

which gives us

$$\lim_{n\to\infty} \sup \int f_n \leq \int f$$

forcing

$$\lim_{n\to\infty} \int f_n = \int f$$

as needed. $\square$

<back2top>

5. Application of MCT for sets

Let $K \subset \mathbb{R}^d$ be a compact subset. And let $O_n = \{x \in \mathbb{R}^d : d(x, K) < \frac{1}{n}\}$. Prove that

$$\lim_{n\to\infty} m(O_n) = m(K).$$

Proof. First note that the O_n are a decreasing sequence. That is,

$$O_n \supset O_{n+1}.$$

holds for every $n \in \mathbb{N}$. And clearly the measure of O_1 is finite thus we can invoke MCT for sets. Thus we get

$$\lim_{n\to\infty} m(O_n) = m(\bigcap_{n\in\mathbb{N}} O_n)$$

Thus is suffices to show

$$m(\bigcap_{n\in\mathbb{N}} O_n) = m(K).$$

So if we can show $K = \bigcap_{n\in\mathbb{N}} O_n$ we are done. Clearny for each n we have $K \subset O_n$ and thus

$$K \subset \bigcap_{n\in\mathbb{N}} O_n$$

holds. To show the other containment suppose towards a contradiction there exists $p \in \bigcap_{n\in\mathbb{N}} O_n$ such that $p \notin K$. Then there exists a sequence $\{y_k\}$ in K that converges to p making p a limit point of K. But K is closed and bounded since it is a compact subset of $\mathbb{R}^d$ thus it contains its limit points forcing $p \in K$ and we are done. □

<back2top>

6. Construction of closed set.

Let $E = [0,1] \setminus \mathbb{Q}$. For any given $\epsilon > 0$ construct a closed subset $F \subset E$ such that

$$m(E \setminus F) \leq \epsilon.$$

Proof. Let $\{r_k\}_{k\in}$ be an enumeration of the rationals in E. We can then define

$$F := [0,1] \setminus \bigcup_{k\in\mathbb{N}} (r_k - \frac{\epsilon}{2^{k+1}}, r_k - \frac{\epsilon}{2^{k+1}}).$$

It is clear to see that $F \subset E$ and that $E \setminus F$ is open and so F is closed in E. We can thus compute

$$\begin{aligned} m(E \setminus F) &= m(\bigcup_{k\in\mathbb{N}} (r_k - \frac{\epsilon}{2^{k+1}}, r_k + \frac{\epsilon}{2^{k+1}}) \setminus \mathbb{Q}) \\ &= m(\bigcup_{k\in\mathbb{N}} (r_k - \frac{\epsilon}{2^{k+1}}, r_k + \frac{\epsilon}{2^{k+1}})) - m(\mathbb{Q}) \\ &= m(\bigcup_{k\in\mathbb{N}} (r_k - \frac{\epsilon}{2^{k+1}}, r_k + \frac{\epsilon}{2^{k+1}})) \\ &= \sum_{k\in\mathbb{N}} |r_k + \frac{\epsilon}{2^{k+1}} - r_k + \frac{\epsilon}{2^{k+1}}| \\ &< \epsilon \end{aligned}$$

as needed. □

<back2top>

7. Bounded Convergence Theorem from Egorov's Theorem

Prove the BCT via Egorov's Theorem. First we state the two major results involved.

> Egorov's Theorem: Let $\{f_n\}$ be a sequence of functions. If they are measurable on a set E of finite measure and $f_n \to f$, then for any $\epsilon > 0$ there exists a closed set $A_\epsilon \subset E$ such that
>
> $$f_n \xrightarrow{\text{uniform}} f$$
>
> on A_ϵ and
>
> $$m(E \setminus A_\epsilon) \leq \epsilon.$$
>
> BCT: Let $\{f_n\}$ be a sequence of functions. If they are bounded, measurable, and supported on a set E of finite measure and if $f_n \to f$, then f is also bounded measurable and supported on E and we have that
>
> $$\int |f_n - f| \to 0.$$
>
> And consequently,
>
> $$\int f_n \to \int f$$
>
> as n tends to ∞.

Proof. So in the spirit of proving the Bounded Convergence Theorem let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions. Assume they are bounded, measurable and supported on E where $m(E) < \infty$. Furthermore suppose we have

$$f_n \to f.$$

Note that $f_n(x) = 0$ for every $x \notin E$ and there is some $M \in \mathbb{R}$ such that

$$|f_n(x)| \leq M$$

for every $n \in \mathbb{N}$ which we get from boundedness. As the f_n are supported on a set of finite measure say E, they are defined on E as well thus we can invoke Egorov. Let $\epsilon > 0$ be given. We are (By Egorov) guaranteed the existence of

$$A_\epsilon \subset E$$

closed such that

$$f_n \xrightarrow{\text{uniform}} f$$

on A_ϵ and

$$m(E \setminus A_\epsilon) \leq \epsilon.$$

Thus as $n \to \infty$, for every $x \in A_\epsilon$ we have that

$$|f_n(x) - f(x)| \leq \epsilon.$$

We can thus compute

$$\begin{aligned}\int_E |f_n - f| &\leq \int_{A_\epsilon} |f_n - f| + \int_{E \setminus A_\epsilon} |f_n - f| \\ &\leq \epsilon m(E) + 2Mm(E \setminus A_\epsilon) \\ &< \epsilon m(E) + 2M\epsilon \\ &= \epsilon(m(E) + 2M)\end{aligned}$$

As ϵ was arbitrary, we are done. □

<back2top>

8. Approximation via Closed and Compact (sub)sets

If $E \subset \mathbb{R}^d$ is Lebesgue measurable, prove you can approximate it via closed and compact subsets.

Proof. First we note that $E \subset \mathbb{R}^d$ is Lebesgue measurable if for any $\epsilon > 0$ there exists $O \in \tau_{\mathbb{R}}$ with $O \supset E$ such that

$$m(O \setminus E) \leq \epsilon.$$

(Here m denotes the Lebesgue measure). Furthermore note complements, countable union and intersections (of measurable) are measurable as well.
We begin with proving if E is Lebesgue measurable, then for any $\epsilon > 0$ there exists a closed set $F \subset E$ such that

$$m(E \setminus F) \leq \epsilon.$$

As E is measurable (element of the sigma algebra) then so is E^c. Then for any given $\epsilon > 0$ there exists an open set $O \supset E^c$ such that

$$m(O \setminus E^c) \leq \epsilon.$$

Note that O^c is closed by definition since $O \in \tau_{\mathbb{R}}$ and furthermore we know

$$O^c \subset E.$$

Then it is clear that $O \setminus E^c = E \setminus O^c$ as sets and thus

$$\begin{aligned} m(E \setminus O^c) &= m(O \setminus E^c) \\ &\leq \epsilon. \end{aligned}$$

as needed.
Lastly, we show that if E is measurable with $m(E) < \infty$, then we can approximate E via compact subset. That is, for any $\epsilon > 0$ we are guaranteed the existence of $K \subset E$ compact such that

$$m(E \setminus K) \leq \epsilon.$$

Let $\epsilon > 0$ be given. Note we can find a closed subset $F \subset E$ such that

$$m(E \setminus F) \leq \frac{\epsilon}{2}.$$

For every $n \in \mathbb{N}$ we can define

$$K_n := F \cap \overline{B_n((0,0))}.$$

Then the K_n are an increasing sequence of set forcing $E \setminus K_n$ to be decreasing. In fact, they decrease down to $E \setminus F$ by MCT for sets. Then we can compute

$$\begin{aligned} \lim_{n\to\infty} m(E \setminus K_n) &= m(\bigcap E \setminus K_n) \\ &= m(E \setminus F) \\ &\leq \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

As ϵ was arbitrary we are done. To really complete this we mention why K_n is compact for each $n \in \mathbb{N}$. Note for each $n \in \mathbb{N}$ we have that

$$K_n \subset B_n((0,0)).$$

Thus K_n is bounded for each n. To see the K_n are all closed, note that for each n, K_n is the intersection of two closed sets and thus closed. □

<back2top>

9. Determine the integral

Determine what integral this is

$$\lim_{n\to\infty}\int_6^n (1+\frac{x}{n})^n.$$

Proof. We will appeal to the DCT. First note that

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x.$$

Then we can define $f_n := (1+\frac{x}{n})^n e^{-2x}$ and it is pretty clear to see that

$$f_n \to \chi_{[6,\infty)}e^{-x}$$

a.e., then the integrable function g can be defined as $g(x) = \chi_{[0,\infty)}e^{-x}$ and we clearly have that $g(x)$ is an upper bound for the f_n thus by the DCT, the given integral converges to the integral of $\chi_{[6,\infty)}e^{-x}$, that is,

$$\lim_{n\to\infty}\int_6^n (1+\frac{x}{n})^n e^{-2x}dx = \int_6^\infty e^{-x}dx$$

and the RHS here is just e^{-6}. □

<back2top>

10. Countable Sub-Additivity

Prove countable sub-additivity for outer measure.

Proof. Let $E_1, E_2, ..$ be a sequence of sets. If they are all measurable, then we prove that

$$m_*(\bigcup_k E_k) \leq \sum_k m_*(E_k).$$

First, if there is some $k \in \mathbb{N}$ such that

$$m_*(E_k) = \infty,$$

it becomes trivial. So let us assume that for every $k \in \mathbb{N}$ we have

$$m_*(E_k) < \infty.$$

Let $\epsilon > 0$ be given. For each $k \in \mathbb{N}$ we can find a countable cover $\{Q_k^j\}_j$ of E_k with $E_k \subset \bigcup_j Q_j^k$ such that

$$\sum_j Q_j^k \leq m_*(E_k) + \frac{\epsilon}{2^k}.$$

Thus we have the containment

$$\bigcup_k E_k \subset \bigcup_{k,j} Q_j^k.$$

And thus we can compute

$$\begin{aligned} m_*(\bigcup_k E_k) &\leq \sum_{k,j} |Q_j^k| \\ &= \sum_j \sum_k |Q_j^k| \\ &\leq \sum_k (m_*(E_k) + \frac{\epsilon}{2^k}) \\ &= \sum_k m_*(E_k) + \epsilon \end{aligned}$$

and ϵ was arbitrary thus we are done. □

<back2top>

11. Lipschitz preserves continuity

If f is Lipschitz continuous over some measure zero set E, then $m(f(E)) = 0$.

Proof. Let

$$f : \mathbb{R} \to \mathbb{R}$$

be a map. Assume f is Lipschitz with Lipschitz constant k and $E \subset \mathbb{R}$ has measure zero. That is, for every $x, y \in \mathbb{R}$ we have

$$|f(y) - f(x)| \leq k|y - x|$$

and we know $m(E) = 0$. Since $m(E) = 0$, for any $\epsilon > 0$ there exists a collection of radius r_j balls say $\{B_{r_j}\}_{j \in \mathbb{N}}$ with

$$E \subset \bigcup_j B_{r_j}$$

such that

$$\sum_j m(B_{r_j}) < \epsilon.$$

By the Lipschitz property for each $j \in \mathbb{N}$, $f(B_{r_j})$ is contained in a ball of radius kr_j. Thus we have

$$m(f(B_{r_j})) \leq km(B_{r_j}).$$

We can now compute

$$\begin{aligned} m(f(E)) &\leq k \sum_{j \in \mathbb{N}} m(B_{r_j}) \\ &< k\epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have that $m(f(E)) = 0$ as needed. □

<back2top>

12. Construction of closed subset

Find a closed set in $[0,1] \setminus \mathbb{Q}$ within ϵ size away.

Proof. Let $E = [0,1] \setminus \mathbb{Q}$ and $\epsilon > 0$ be given. Take $\{r_k\}$ to be an enumeration of $\mathbb{Q} \cap [0,1]$. Then we can define our closed subset of E to be

$$F := [0,1] \setminus \bigcup_{k \in \mathbb{N}} (r_k - \frac{\epsilon}{2^{k+2}}, r_k + \frac{\epsilon}{2^{k+2}}).$$

Clearly we have $F \subset E$. Since F^c is open in E, F is closed. We can now compute

$$\begin{aligned} m(E \setminus F) &= m(\bigcup_{k \in \mathbb{N}} (r_k - \frac{\epsilon}{2^{k+2}}, r_k + \frac{\epsilon}{2^{k+2}}) \setminus \mathbb{Q}) \\ &= m(\bigcup_{k \in \mathbb{N}} (r_k - \frac{\epsilon}{2^{k+2}}, r_k + \frac{\epsilon}{2^{k+2}})) - m(\mathbb{Q}) \\ &= m(\bigcup_{k \in \mathbb{N}} (r_k - \frac{\epsilon}{2^{k+2}}, r_k + \frac{\epsilon}{2^{k+2}})) \\ &= \sum_{k \in \mathbb{N}} |r_k + \frac{\epsilon}{2^{k+2}} - r_k + \frac{\epsilon}{2^{k+2}}| \\ &= \sum_{k \in \mathbb{N}} \frac{\epsilon}{2^{k+1}} \\ &= \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

and ϵ was arbitrary so we are done. □

<back2top>

13. Tchebychev inequality

Let f be a function. Assume f is integrable and non-negative on some Lebesgue measurable set E. Here measure is given by m. For $\alpha > 0$ define

$$E_\alpha := \{x \in E : f(x) > \alpha\}.$$

Prove then that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int_E f dm.$$

Proof. First we define the simple function

$$g(x) = \alpha \chi_{E_\alpha}(x).$$

As f is defined on all of E where as g is defined over E_α together with zero product property we have that

$$0 \leq g(x) \leq f(x).$$

Thus by monotonicity of the Lebesgue integral we have that

$$\begin{aligned} \alpha m(E_\alpha) &= \int_{E_\alpha} g(x) dm \\ &\leq \int_E f(x) dm \end{aligned}$$

And as α is non-zero we can divide by it and we get the desired result.
Furthermore, assuming $\int_E f = 0$ we can use the inequality to show $f = 0$ a.e. on E. Take $\alpha = \frac{1}{n}$ for some $n \in \mathbb{N}$. Next compute the measure of $E_{\frac{1}{n}}$ using the inequality.

$$\begin{aligned} m(E_{\frac{1}{n}}) &\leq \frac{1}{\frac{1}{n}} \int_E f dm \\ &= n \int_E f dm \\ &= n(0) \\ &= 0 \end{aligned}$$

Thus $m(E_{\frac{1}{n}}) = 0$. Before we continue note that $\bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = E_0$. And we can compute

$$\begin{aligned} m(E_0) &= m(\{x \in E : f(x) > 0\}) \\ &= m(\bigcup_{n \in \mathbb{N}} \{x \in E : f(x) > \frac{1}{n}\}), \end{aligned}$$

which is a countable union of measure zero sets thus by countable sub-additivity we have that $m(E_0) = 0$. This gives us that for non-negative f, we have $f = 0$ a.e. on E. □

<back2top>

14. Point-wise convergence

Prove that point-wise convergence implies convergence in measure.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions. Assume that the f_n are measurable and finite valued on a finite measure set E. If in addition $f_n \to f$, then we prove that

$$f_n \xrightarrow{\text{measure}} f.$$

That is, for any $\epsilon > 0$ we show

$$m(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$$

as $n \to \infty$. As the f_n are measurable on a finite measure set and $f_n \to f$ point-wise a.e. on E, we can invoke Egorov's Theorem which says for any $\epsilon > 0$ there exists $A_\epsilon \subset E$ closed such that

$$m(E \setminus A_\epsilon) \leq \epsilon,$$

and

$$f_n \xrightarrow{\text{uniform}} f$$

on A_ϵ. So for any $x \in A_\epsilon$ we know

$$|f_n(x) - f| \leq \epsilon.$$

This gives us that $\{x : |f_n(x) - f(x)| > \epsilon\} \subset E \setminus A_\epsilon$ then by monotonicity,

$$m(\{x \in E : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$$

as needed.
Furthermore, convergence in L^1 implies convergence in measure. Let us assume then that $f_n \to f$ in L^1. That is,

$$\int |f_n - f| \to 0$$

as n tends to ∞. Let $\epsilon > 0$ be given, by the infamous inequality we get that

$$\begin{aligned} m(\{x \in E : |f_n(x) - f(x)| > \epsilon\}) &\leq \frac{1}{\epsilon} \int_E |f_n(x) - f(x)| \\ &= 0 \end{aligned}$$

and we are done. ∎

<back2top>

15. Continuous functions are Borel

Prove continuous functions are Borel-Measurable.

Proof. Let
$$g : \mathbb{R}^n \to \mathbb{R}$$
be a function. If g is continuous, prove g is Borel measurable by showing that given

$$G = \{U \subset \mathbb{R} : g^{-1}(U) \text{is Borel}\},$$

open sets are in G and that G forms a σ-algebra. To see that open sets are in G let $V \subset \mathbb{R}$ be open. Then since g is continuous, $g^{-1}(V) \subset \mathbb{R}^n$ is open, and in particular a Borel set in $\mathbb{R}^n$ thus $V \in G$ as needed.

Next, we show G satisfies the axioms of a σ-algebra. Let $U_1, U_2, ... \in G$. Then for each $j = 1, 2, ..$ we have that $U_j \subset \mathbb{R}$ and $g^{-1}(U_j)$ is Borel. We would like to show

$$\bigcup_{j \in \mathbb{N}} U_j, \bigcap_{j \in \mathbb{N}} U_j \in G.$$

But then we have
$$g^{-1}(\bigcup_{j \in \mathbb{N}} U_j) = \bigcup_{j \in \mathbb{N}} g^{-1}(U_j)$$
and
$$g^{-1}(\bigcap_{j \in \mathbb{N}} U_j) = \bigcap_{j \in \mathbb{N}} g^{-1}(U_j)$$
which are Borel as well since Borel sets are closed under countable union and intersections. Lastly we show G is closed under complements. Let $U \in G$, then $g^{-1}(U)$ is Borel. We want to show $U^c \in G$ which means $g^{-1}(U^c)$ is Borel. However, as $U \subset \mathbb{R}$ we know that

$$\begin{aligned} g^{-1}(U^c) &= g^{-1}(\mathbb{R} \setminus U) \\ &= g^{-1}(\mathbb{R}) \setminus g^{-1}(U) \\ &= \mathbb{R}^n \setminus g^{-1}(U) \\ &= g^{-1}(U)^c \end{aligned}$$

which is Borel as complement of Borel is Borel thus G forms a σ-algebra. □

<back2top>

16. Show $C([-1,1])$ is not complete.

Show $C([-1,1])$ is not complete in this norm

$$||f|| = \int_{-1}^{-1} |f(x)|dx.$$

Proof. Consider the sequence of functions

$$f_n(x) = \{ \begin{array}{cc} 1 & x < \frac{1}{2} \\ 0 & x > \frac{1}{2} + \frac{1}{n} \end{array} .$$

Then f is continuous a.e. and divergent even though it is Cauchy since max distance is 1. ☐

<back2top>

17. Construct sequence converging in L^p and m but not a.e. (& MORE)

Proof. We construct the infamous tpywriter sequence which converges both in measure and in $L^p(\mathbb{R})$ but not a.e. Consider

$$f_n(x) = \chi_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}(x)$$

for every $k \geq 0$ such that $n \in [2^k, 2^{k+1})$. Then we have

$$f_n(x) \xrightarrow{\text{measure}} 0, f_n(x) \xrightarrow{L^1(\mathbb{R})} 0.$$

But for every $x \in (0,1)$, $f_n(x)$ does not converge a.e.

Some additional examples:
The sequence f_n defined via

$$f_n(x) := \chi_{[n,n+1]}(x)$$

converges to 0 pointwise but in no other sense.
The sequence f_n defined via

$$f_n(x) := \frac{\chi_{[0,n]}(x)}{n}$$

converges to 0 uniformly and in measure but not in L^1.
The sequence f_n defined via

$$f_n(x) := n\chi_{[\frac{1}{n}, \frac{2}{n}]}(x)$$

converges to 0 pointwise but not uniform or in any norm. □

<back2top>

18. Cauchy sequence of simple functions

Let $E \subset \mathbb{R}^d$ and f_n be simple functions on E. If E has finite measure and the f_n are bounded with $f_n(x) \to f(x)$ for every $x \in E$, then prove

$$\lim_{n\to\infty} \int f_n$$

is Cauchy.

Proof. Let $\epsilon > 0$ and $E \subset \mathbb{R}^d$ be given. Assume $m(E) < \infty$, then since simple functions are integrable, we can apply Egorov's Theorem and so there exists a closed set $A_\epsilon \subset E$ such that

$$f_n(x) \xrightarrow{\text{uniform}} f(x)$$

for every $x \in A_\epsilon$ and $m(E \setminus A_\epsilon) < \epsilon$. Then we have

$$\begin{aligned}
\left| \int_E f_n(x) - f_m(x) dx \right| &\leq \int_E \left| f_n(x) - f_m(x) \right| dx && \text{triangle inequality} \\
&= \int_{A_\epsilon} \left| f_n(x) - f_m(x) \right| + \int_{E \setminus A_\epsilon} \left| f_n(x) - f_m(x) \right| && \text{additivity} \\
&\leq \int_{A_\epsilon} \left| f_n(x) - f_m(x) \right| + 2Mm(E \setminus A_\epsilon) && \text{boundedness of the } f_n \\
&\leq \int_{A_\epsilon} \left| f_n(x) - f_m(x) \right| + 2M\epsilon && \text{By Egorov as } m(E \setminus A_\epsilon) \leq \epsilon \\
&\leq m(E)\epsilon + 2M\epsilon && \text{Egorov as } f_n(x) \xrightarrow{\text{uniform}} f(x) \text{ on } A_\epsilon
\end{aligned}$$

As ϵ was arbitrary and $m(E) < \infty$ we are done. □

<back2top>

19. Existence of simple going to f bounded on $\mathbb{R}^d$

Let f be a function on $\mathbb{R}^d$. If f is measurable and $f(x) \in [0, M]$ for some $M \in \mathbb{R}^+$, then show there exists a sequence $\{f_n\}$ of simple functions with $f_n(x) \in [0, M]$ for every n, such that

$$f_n \xrightarrow{\text{uniform}} f$$

Proof. Consider the sequence of functions defined via

$$f_n(x) := \sum_{k=1}^{\lfloor 2^n M \rfloor + 1} \frac{k-1}{2^n} \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n})}.$$

Then $\delta = \frac{1}{2^n}$ gives us

$$f_n \xrightarrow{\text{uniform}} f$$

as needed. □

<back2top>

20. Examples

Let $f : [0, 1] \to \mathbb{R}$. Give function that is Lebesgue integrable but not Riemann. And one that such that f is Lebesgue Integrable but f^2 is not.

Proof. For a Lebesgue integrable but non-Riemann integrable let

$$f : [0, 1] \to \mathbb{R}$$

be a function defined via

$$x \mapsto \left\{ \begin{array}{ll} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{array} \right. .$$

Then f is not Riemann integrable and has Lebesgue integral value of 0.
On the other hand for a function $f : [0, 1] \to \mathbb{R}$ such that f^2 is not Lebesgue integrable consider

$$f(x) = x$$

And then

$$f^2(x) = x^2$$

which is not Lebesgue integrable over $[0, 1]$. □

<back2top>

21. Approximate finite measure set with open

Let $E \subset \mathbb{R}^d$. If E has finite measure, then show for any $\epsilon > 0$ there exists an open set $O \supset E$ such that

$$m_*(O \setminus E) \leq \epsilon.$$

Proof. Let $\epsilon > 0$ be given. As E has finite measure, there exists a collection of closed cubes $\{Q_j\}_{j \in \mathbb{N}}$ with

$$E \subset \bigcup_{j \in \mathbb{N}} Q_j$$

such that

$$\sum_{j \in \mathbb{N}} |Q_j| \leq |E| + \frac{\epsilon}{2}.$$

For each j, take Q_j^* to be a closed cube with

$$Q_j \subset \text{int} Q_j^*$$

such that

$$|Q_j^*| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}.$$

Then take O to be $\bigcup_{j \in \mathbb{N}} \text{int} Q_j^*$ we can compute

$$\begin{aligned} m_*(O) &= m_*(\bigcup_{j \in \mathbb{N}} \text{int} Q_j^*) \\ &\leq \sum_{j \in \mathbb{N}} |\text{int} Q_j^*| \\ &\leq \sum_{j \in \mathbb{N}} |Q_j^*| \\ &\leq \sum_{j \in \mathbb{N}} |Q_j| + \frac{\epsilon}{2^{j+1}} \\ &\leq |E| + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= m_*(E) + \epsilon \end{aligned}$$

and thus $m_*(O \setminus E) \leq \epsilon$ as needed. □

<back2top>

22. Prove DCT using Fatou's.

Prove the Dominated Convergence Theorem via Fatou's Lemma.

Proof. We first state the two.

> Fatou's Lenmma: Let $\{f_n\}$ be a sequence of functions. If the f_n are non-negative, measurable, and $f_n \to f$ point-wise almost everywhere, then
>
> $$\int f \leq \liminf \int f_n.$$
>
> DCT: Let $\{f_n\}$ be a sequence of functions. If the f_n are measurable and there exists an integrable function g such that
>
> $$|f_n| \leq g$$
>
> and $f_n \to f$ p.w.a.e., then
>
> $$\int |f_n - f| \to 0$$
>
> as $n \to \infty$.

So let $\{f_n\}$ be a sequence of functions. Suppose that the f_n are measurable and $f_n \to f$. Furthermore, suppose there exists an integrable function g such that for every n,

$$|f_n| \leq g.$$

Since $f_n \to f$, then we also have

$$|f| \leq g.$$

Then

$$g \pm f_n \to g \pm f$$

and

$$g \pm f_n \geq 0.$$

We first apply Fatou's Lemma to $g + f_n$ as it is non-negative. So we compute

$$\begin{aligned} \int g + \int f &\leq \liminf_n \Big(\int g + \int f_n \Big) \\ &= \int g + \liminf_n \int f_n. \end{aligned}$$

As g is integrable we have

$$\int f \leq \liminf_n \int f_n.$$

Similarly we have

$$\begin{aligned} \int g - \int f &\leq \liminf_n \Big(\int g - \int f_n \Big) \\ &= \int g - \liminf_n \int f_n \end{aligned}$$

Then dividing by (-1) and since g is integrable we get

$$\int f \geq \limsup_n \int f_n$$

And we are done. $\square$

<*back2top*>

23. If $f \in L^1$, then $f_h - f \in L^1$

Prove that if $f \in L^1$ then so is $f_h - f$.

Proof. Let $\epsilon > 0$ and $f \in L^1(\mathbb{R}^d)$. We know there exists (by density in $L^1(\mathbb{R}^d)$) a continuous function of bounded support g such that for any $\epsilon > 0$,

$$||f - g|| < \epsilon.$$

Then we can write

$$\begin{aligned}
||f_h - f||_{L^1} &= ||f - g|| + ||g - g_h|| + ||g_h - f|| && \text{triangle inequality} \\
&= ||f - g|| + ||g - g_h|| + ||(f - g)_h|| && \text{symmetry, and } ||f||_{L^1} = ||f_h||_{L^1} \\
&= 2||f - g|| + ||g - g_h|| && \text{since } ||f||_{L^1} = ||f_h||_{L^1} \\
&< 2\epsilon + ||g - g_h|| && \text{assumption, } g \text{ is dense.} \\
&= 2\epsilon && \text{as } g \text{ is continuous of bounded support}
\end{aligned}$$

As ϵ was arbitrary we are done. □

<back2top>