Math 566 Project: Analytical alternative to a Riemann surface- Topics in Complex Analysis ; Riemann Surfaces

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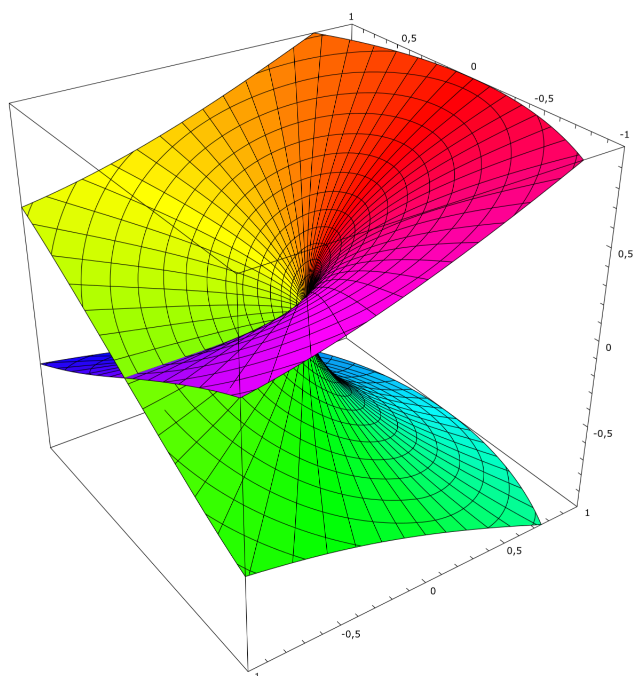
Abstract

We have seen from early in the semester this notion of Analytic continuation where you take a given analytic function and extend its domain to where it is analytic on this new domain, recall Exercise 2. This method was what first motivated the theory of Riemann surfaces as we take 'patches' of the complex plane so to speak. More recently however, modern techniques and axioms from manifold theory have been used to develop the theory from more topological and less analytic prospective which is more or less what he have focused on in the scope of our course. In this paper we aim to give a formal introduction to the construction of a Riemann surface via (maximal) analytic continuation. We begin with introductin the notion with a familiar example, then generalizing and compare contrasting the two approaches.



0. What is analytic continuation

From 562, we all saw how we were able to extend the natural log from high school to take on complex values. That is, in high school we had

$$f(x) = \log x$$

Where $x \in (0, \infty) = \mathbb{R}^+$.

Early in this class we defined a function element as an ordered pair

$$(f, D)$$

to be an analytic function defined on some open and connected subset of $\mathbb{C}$.

In this language of function elements, we can let

$$(f_0, D_0) := (\log x, \mathbb{R}^+)$$

And

$$(f_1, D_1) := (\log|z| + 2\pi ik, \mathbb{C}\setminus\{z \in \mathbb{R} : z \leq 0\})$$

Here we denoted f_1 by Log z and defined this as the *principle branch* of the log

Then (f_1, D_1) is an analytic continuation of (f_0, D_0) along .

Before moving on, let us make this definition a bit more formal which requires a bit of work.

1. Construction of continutation via chains

We are all familiar with continuation along a curve, but we show how you obtain this from *chain* of disks.

Def 1.1 Given two function elements $(f_0, D_0), (f_1, D_1)$, we say they are *direct continutations* of eachother given that

$$\text{(i)} D_0 \cap D_1 \neq \emptyset$$
$$\text{(ii)} f_0(z) = f_1(z) \quad , \forall z \in D_0 \cap D_1$$

hold, and we write

$$(f_0, D_0) \quad (f_1, D_1)$$

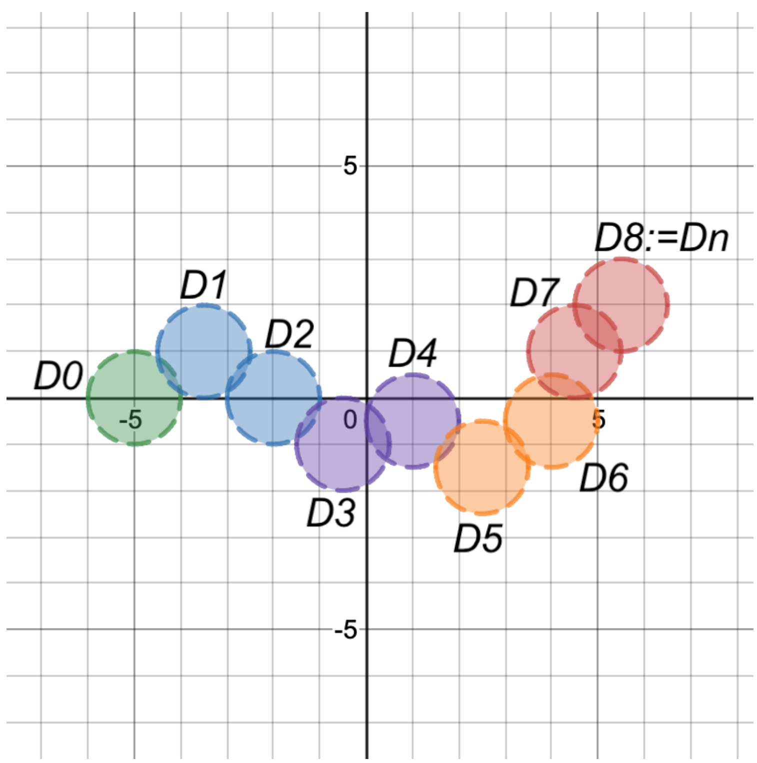
Def 1.2 A *chain* is a finite sequence C of disks say

$$C = \{D_0, D_1, \ldots, D_n\}$$

$\ni$

$$D_{i-1} \cap D_i \neq \emptyset$$

$\forall i = 1, \ldots, n$, see figure:



Def 1.3 For a given function element say

$$(f_0; D_0)$$

if there is a collection of function elements

$$\{(f_i; D_i)\}_{i=1}^{n}$$

$\ni$

$$(f_{i-1}; D_{i-1}) \quad (f_i; D_i)$$

$\forall i = 1; \ldots; n$, then we say $(f_n; D_n)$ is the *analytic continuation along* C.

Note that if such a function element exists, it is uniquely determined by f_0 and the chain C

Lemma 1.1 In the above setting, f_n is uniquely determined.

Pf.

Let $(f_0; D_0)$ be a function element and

$$C = \{D_0; D_1; \ldots; D_n\}$$

be a chain with

$$\{(f_i; D_i)\}_{i=1}^{n}$$

a collection of function elements $\ni$ $(f_n; D_n)$ is the analytic continuation along C.

Then the function elements are pairwise direct continutations of one another, namely we have

$$(f_0; D_0) \quad (f_1; D_1)$$

Then they agree on the intersection, that is

$$f_0(z) = f_1(z) \quad ; \forall z \in D_0 \cap D_1$$

We are aiming to show uniqueness so suppose we also have

$$(f_0; D_0) \quad (\tilde{f}_1; D_1)$$

For some analytic function $\tilde{f}_1$, then we have

$$f_0(z) = \tilde{f}_1(z) \quad ; \forall z \in D_0 \cap D_1$$

which implies

$$f_1(z) = \tilde{f}_1(z) \quad ; \forall z \in D_0 \cap D_1$$

And by connectedness of the D_i we in particular have that

$$f_1(z) = \tilde{f_1}(z) \quad ; \forall z \in D_1$$

thus by induction on $|C|$ we have uniqeness

Note that the relation is not an equivalence relation as it does not obey transitivity, we saw an example of this in Exercise 2.

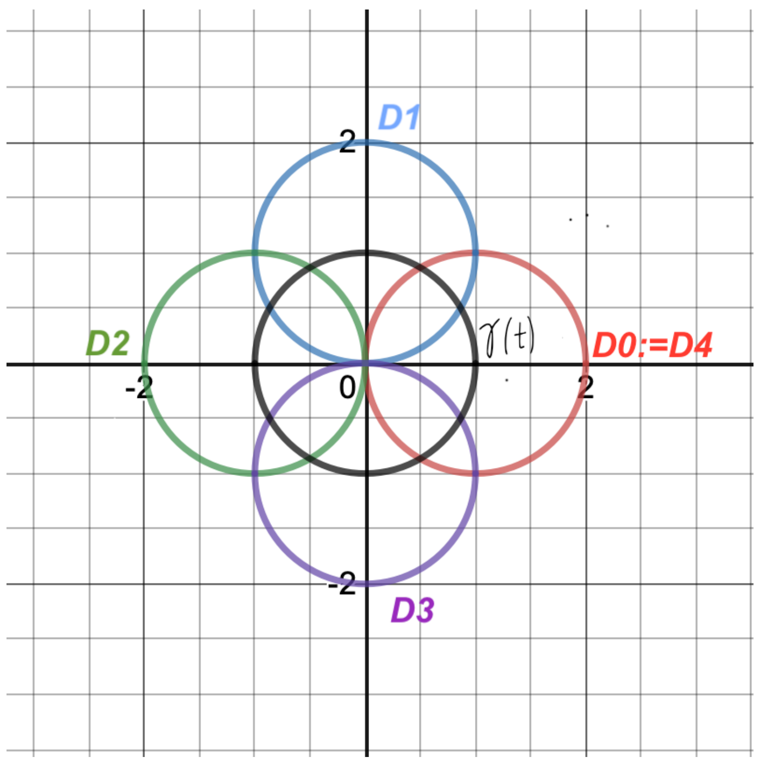
In fact in this context of chains, it is interesting to note that in Exercise 2 we had a chain, namely

$$\mathsf{C} = \{B_1(1); B_1(i); B_1(-1); B_1(-i)\}$$

even though we were asked to analytically continue the function along paths

$$\gamma(t) = \exp(2\pi i t); \gamma(t) = \exp(4\pi i t) \quad ; \forall t \in [0; 1]$$

See the figure:



In fact the notion of analytically continuing along a path is equivalent to analytically continuing along a chain (Exercise 1 Ch IX.1, Conway). If we suppose elements of C are pairwise disjoint is transitive here.

2. Traveling back along the path

Def 2.1 We say a given chain

$$C = \{D_0, D_1, \dots, D_n\}$$

covers a curve γ with domain $[0, 1]$, if there are numbers

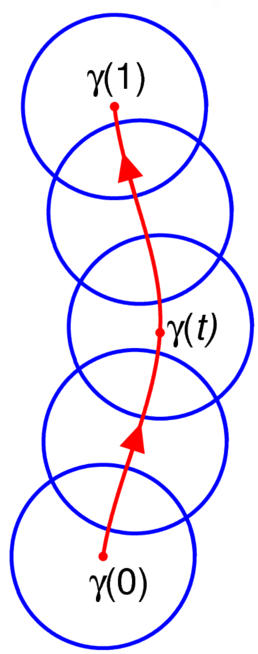
$$0 = t_0 < t_1 < \dots < t_n = 1$$

$\ni$ D_0 is centered at $\gamma(0)$ and D_n is centered at $\gamma(1)$ and

$$\gamma([t_i, t_{i+1}]) \subset D_i \quad ; \forall i = 0, 1, \dots, n-1$$

If (f_0, D_0) can be analytically continued to (f_n, D_n) along this chain, then (f_n, D_n) is the *analytic continutation along* γ.

We note three things, one is that if a function element can admit at most one analytic continutation (which we showed is uniqie) we leave the proof to the interested reader. Hint: Uses the fact that transitivity holds under a 'nice enough' condition), secondly we can naively think of γ as 'running through' the chain as see in the photo, lastly we will refer to (f_n, D_n) as the maximal analytic continuation of (f_0, D_0) along γ.



Back to the language of germs now, we have the following proposition:

Proposition 2.1 Let

$$\gamma : [0, 1] \to \mathbb{C}$$

3

$$(0) = a; \quad (1) = b$$

Let

$$f(f_t; D_t) : t\ 2\ [0; 1]g; f(g_t; E_t) : t\ 2\ [0; 1]g$$

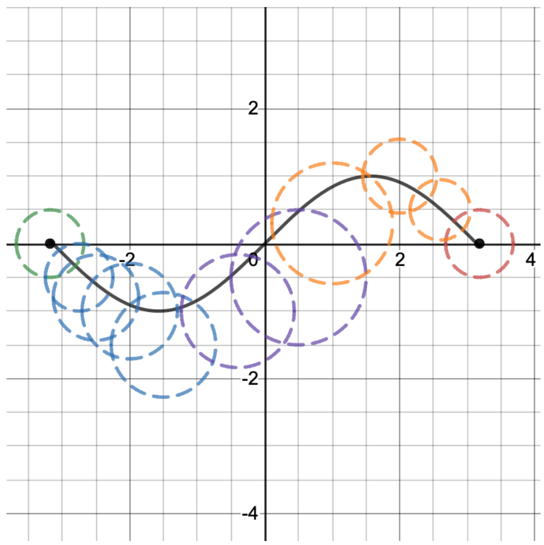
be analytic continutations of a given function element along *3*

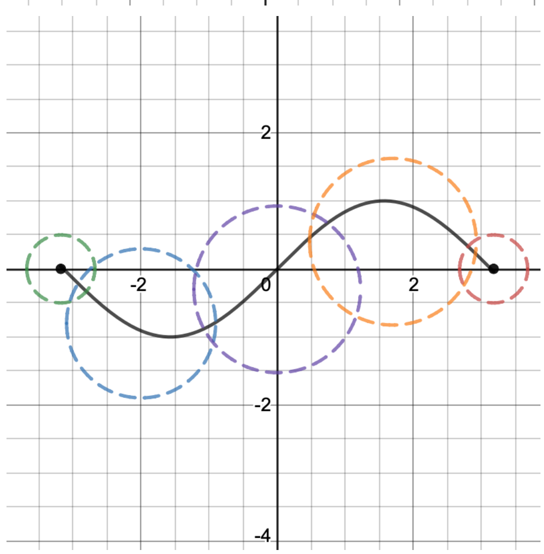
$$[f_0]_a = [g_0]_a$$

Then

$$[f_1]_b = [g_1]_b$$

as portrayed here





Pf.

We aim to show that

$$A := \{t \in [0,1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$$

is both open and closed in $[0,1]$ and use connectedness of $[0,1]$ to conclude A is all of $[0,1]$.

Note that $0 \in A$ implying $A \neq \emptyset$.

First we show A is open by finding an open ball around some fixed element.

First fix $t \in A$. Then by definition of analytic continutation $\exists \delta > 0 \ni$ if

$$|s - t| < \delta$$

then

$$\gamma(s) \in D_t \cap E_t$$

and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$
$$[g_s]_{\gamma(s)} = [g_t]_{\gamma(s)} \qquad (*)$$

Since $D_t, E_t \subseteq \mathbb{C}$ are open, it follows that $D_t \cap E_t$ is open, thus it contains a connected component.

Let

$$C \subseteq D_t \cap E_t$$

be a connected component, i.e., a maximally connected subset $\ni$

$$\gamma(s), \gamma(t) \in C \quad ; \forall s, t \in [0,1]$$

However we assume $t \in A$ hence

$$f_t(z) = g_t(z) \quad ; \forall z \in C$$

So then $\forall \gamma(s) \in D_t \cap E_t$

$$[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$$

And since $|s - t| < \delta$, we can apply $(*)$ to can swap t's for s's:

$$[f_s]_{\gamma(s)} = [g_s]_{\gamma(s)}$$

So we found an open ball of t fully contained in A, that is

$$(t - \delta, t + \delta) \subseteq A$$

Thus A is open.

Now we must show $A \quad [0,1]$ is closed, we do so by picking an arbitrary limit point and showing it is in A.

We pick the same as before, with the same conditions.

Let t be some limit point of A, then by definition $\exists s \in A \ni$

$$|s \quad t| < \qquad (\quad)$$

We want for $t \in A$, that is

$$[f_t]_{(t)} = [g_t]_{(t)}$$

Take an open and connected subset of the intersection $D_t \cap E_t$, call it G, $\ni$

$$((t \quad , t + \quad)) \quad G$$

By ( ) then we can say $(s) \in G$ thus by definition of our A,

$$f_s(z) = g_s(z) \qquad , \forall z \in G$$

Applying ( ) from the proof of open we have that

$$f_s(z) = f_t(z)$$
$$g_s(z) = g_t(z)$$

which clearly implies

$$f_t(z) = g_t(z) \qquad , \forall z \in G$$

But G must contain a limit point in $D_t \cap E_t$ thus

$$[f_t]_{(t)} = [g_t]_{(t)}$$

Giving us that $t \in A$ so it contains its limit points thus it is closed.

Since

$$\emptyset \neq A \quad [0,1]$$

is open and closed and $[0,1]$ is connected, it follows that $A = [0,1]$

Note that we have established one of the key facts concearning the uniqeness of continutation, that is distinct continutations along same path, we will also encounter a version that assumes we have distinct but homotopic paths.

Def 2.2 The germ $[f_1]_b$ is the *analytic continutation* of $[f_0]_a$ along .

Def 2.3 For a given function element (f, D) the *complete analytic continutation obtained from* (f, D) is the collection $\mathcal{F}$ of all germs $[g]_b$ for which $\exists a \in D$ and γ going from a to $b \ni [g]_b$ is the analytic continutation of $[f]_a$ along γ

It is worth noting a few things here. First that choice of a is arbitrary, and also that $[f]_z \in \mathcal{F}$ $\forall z \in D$.

We must establish that our definition of $\mathcal{F}$ is well-defined by formalizing it into a function.

That is, we need a domain and range. Clearly we want it to out put complex values.

It turns out we can naively let $\mathcal{F}$ be its own domain, i.e.,

$$\mathcal{R} := \{(x, [f]_z) : [f]_z \in \mathcal{F}\}$$

Then it makes sense for us to define

$$\mathcal{F} : \mathcal{R} \to \mathbb{C}$$

via

$$(z, [f]_z) \mapsto f(z)$$

We leave it to the interested reader to verify this map is well-defined.

Spoiler alert: $\mathcal{R}$ is in some sense going to end up being a Riemann surface which we will see shortly. Before moving onto the next section, it is worth mentioning the big uniqueness theorem for continuation which requires a definition.

Def 2.4 As mentioned previously, we may not always be guaranteed a continuatation which leads us to this definition.

Let (f, D) be a function where $D \subseteq E \subseteq \mathbb{C}$ open and connected, we say (f, D) *admits unrestricted analytic continuatation in* E if for any path γ in E starting in D, $\exists$ an analytic continuation of (f, D) along γ.

Example 2.1

Let $D := B_1(1)$, and f be the principal branch of $\sqrt{z}$.

Then (f, D) admits unrestricted analytic continuation in $\mathbb{C} \setminus \{0\}$ but not in all of $\mathbb{C}$

This leads us to our main uniqueness theorem:

Monodromy Theorem Let (f, D) be a function element and let $D \subseteq E \subseteq \mathbb{C}$ be open and connected $\Rightarrow$ (f, D) admits unrestricted analytic continuation in E.

Let $a \in D, b \in E$ with

$$\gamma_0 : [0, 1] \to E$$
$$\gamma_1 : [0, 1] \to E$$

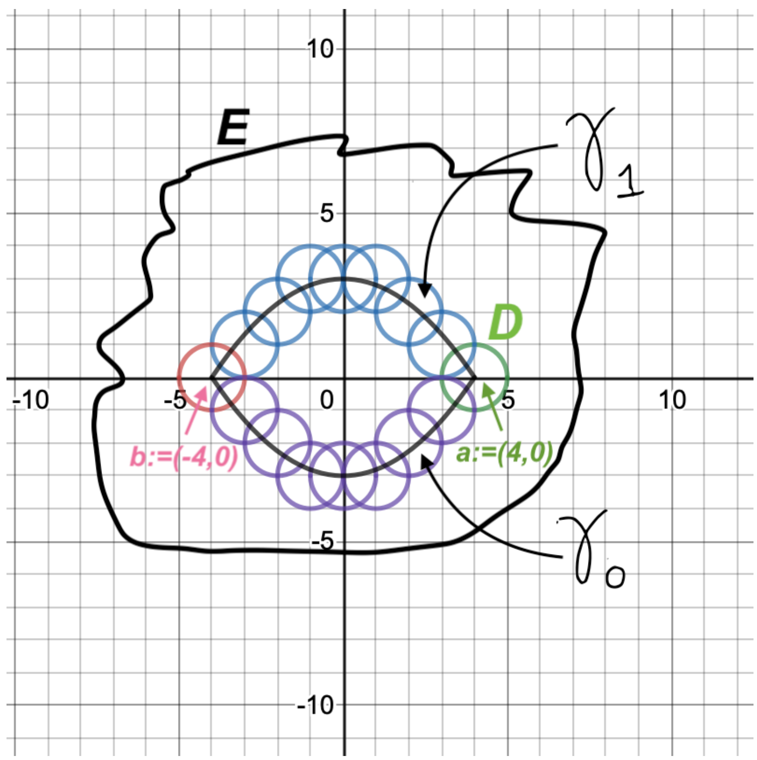
both going from a to b.

Let $\{(f_t, D_t) : t \in [0, 1]\}$, $\{(g_t, B_t) : t \in [0, 1]\}$ be analytic continutations of (f, D) along γ_0, γ_1 resp.

If $\gamma_0 \simeq \gamma_1$ with fixed end points, then

$$[f_1]_b = [g_1]_b$$

That is, the maximal analytic continutations are invariant under choice of path taken up to homotopy. If you look below, it is pretty clear to see $\gamma_0 \simeq \gamma_1$, a depiction of the monodromy.



Pf.

Since γ_0, γ_1 are have fixed end points and are homotopic, $\exists$ a continuous function sending one to the other.

(This is formally referred to as a Homotopoy from one path to the other in Algebraic Topology, we will freely assume the reader has some familiarity with these concepts so we did not bother defining).

Let

$$H : [0,1] \times [0,1] \to E$$

be a continuous function with starting and ending points

$$\begin{aligned} H(t,0) &= \gamma_0(t) \qquad H(t,1) = \gamma_1(t) \\ H(0,u) &= a \qquad H(1,u) = b \end{aligned}$$

$\forall t,u \in [0,1]$.

Now we fix $u \in [0,1]$ and consider

$$\gamma_t(u) := H(t,u)$$

mapping from a to b which admits its own analytic continutation of (f,D) along γ_u call it

$$\{(h_{t,u}, D_{t,u}\}_{t \in [0,1]}$$

so u remains fixed and t ranges over the interval.

By our Proposition 2.1,

$$\begin{aligned} [g_1]_b &= [h_{1,1}]_b \\ [f_1]_b &= [h_{1,0}]_b \end{aligned}$$

for $u = 0,1$ respectively.

We are left to show at this point that the RHS's are equal. To do so we introduce

$$B := \{u \in [0,1] : [h_{1,u}]_b = [h_{1,0}]_b\}$$

And use connectedness again to show $\emptyset \neq B \subset [0,1]$ is open and closed then conclude $B = [0,1]$.

I claim B is in fact nonempty, closed and open in the unit interval forcing it to be everything.

We leave the proof of the claim to the interested reader

Note that we have not formally definied the notion of path equivalence which is called homotopy, for time purposes and since the class is familiar with the notion of a continuous deformation.

3. Build the surface

In order to make R into a Riemann surface, all we have is that it is a graph. We need some sort of *complex structure* on it as well, a way to map it analytically into $\mathbb{C}$ and we require it to be Hausdorff, Second-Countable and connected.

As in the notation above, let $D \subset \mathbb{C}$ be open and connected and let

$$S(D) := \{(z, [f]_z) : z \in D\}$$

And f is analytic at z.

<u>**Def 3.1**</u> Given the projection map

$$\pi : R(D) \to \mathbb{C}$$

via

$$\pi((z, [f]_z)) \mapsto z$$

We say $(S(D), \pi)$ is the *sheaf of germs of analytic functions on* D.

So for any point

$$(z, [f]_z) \in S(D)$$

consider a function element (f_0, D_0) from $[f]_z$, then

$$\forall w \in D_0, \exists (w, [f]_w) \in S(D)$$

Thus for each

$$(z, [f]_z) \in S(D)$$

$\exists$ an associated 'sheet' or surface

$$\{(w, [f]_w) : w \in D_0\}$$

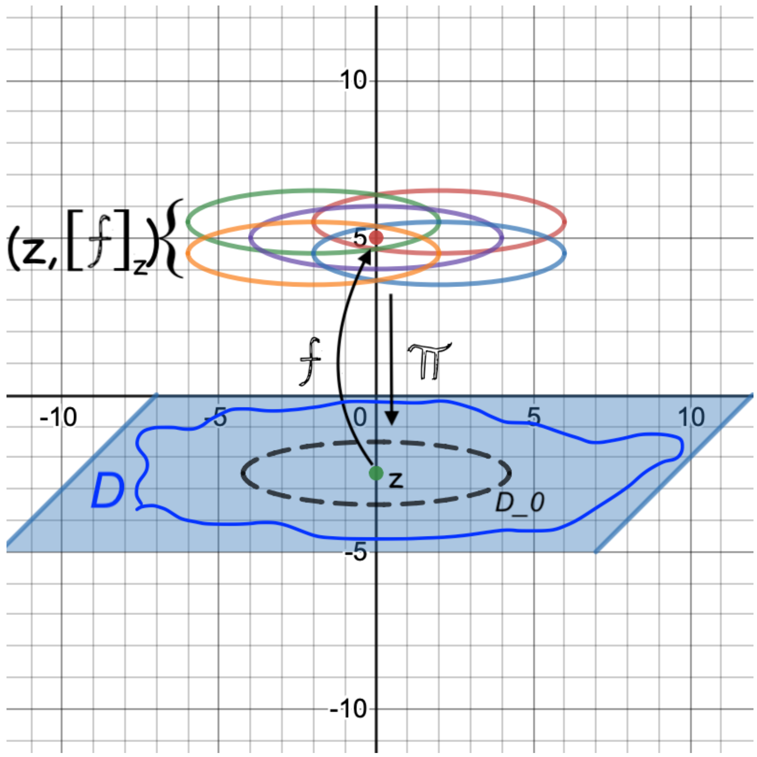
Where we could actually think of each of these as graphs making $R(D)$ just a union of graphs.

This is because elements of $S(D)$ correspond to $(z, f(z)) \in \Gamma[(f, D)]$ where

$$\Gamma(f, D) := \{(x, f(x)) : x \in D\}$$

The key thing here to note is that the surfaces $R(D)$ consists of are not necessarily all disjoint, in fact there is an overlapping which alows us to 'patch' them together.

One way of viewing what is going on is depicted here



We proceed with constructing our Riemann surface.

Let $(f; D)$ be any arbitrary function element and F be the complete analytic function obtained from it. Then we can define

$$R := \{(z, [f]_z) : [f]_z \in F\}$$

which we should note is the maximal connected subset of $S(D)$.

Let

$$\pi : R \to D$$

be the natural projection map defined via

$$(z, [f]_z) \mapsto z$$

Then we claim

$$(R, \pi)$$

turns out to be a Riemann surface. We would first like to show what our open sets look like, suppose we have

$$R = \bigcup_{2A} U$$

Then we have that

$$U$$

4. Different *path* to Riemann surfaces

From class we saw that given a polynomial of degree d, we were able to construct a *locus*,

which is a local graph by Implicit function theorem, or the zero set.

This reduces the degree by 1, And so in the case we have a polynomial F in two variables,

the vanishing space $V(F)$ is a potential candidate to be a Riemann surface iff F is nonsingular.

However, one could also look at the analytic continuation of a given function to construct a Riemann surface.

Let us consider for example the function

$$F(x; y) = y^2 \quad h(x)$$

We have all seen, now we know we can only construct a RS iff $h(x)$ has no repeated roots. (Exercise 14).

5. References

[1] J.B. Conway: Functions of One Complex Variable I, Springer, New York (1978).

[2] W. Rudin: Real and Complex Analysis, McGraw Hill, Chennai, India (1987).

[3] L.V. Ahlfors: Complex Analysis, An introduction to the Theory of One Complex Variable, McGraw Hill, Unites States (1979).